Hence the number of roots of f between a and b is determined from $f, \cdots, f_r$ only. Practically to avoid zeros of f_r, we substitute $a + \epsilon$, $b - \epsilon$ in the first $r + 1$ functions, when ϵ is small. To complete the work it is necessary to discover whether or not any of the roots of f_r are roots of f, and to apply the theorem to all the intervals given by $\pm \infty$ and the roots of f_r.

Example :

$$f(x) = x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1,$$

$$f_1(x) = 5x^4 - 20\,x^3 + 27x^2 - 18x + 5, \quad f_2(x) = x^3 - x.$$

Roots of f_2 are -1, 0, $+1$; of these $+1$ is a root of $f(x)$.

	$-\infty$	$-1-\varepsilon$	$-1+\varepsilon$	$-\varepsilon$	$+\varepsilon$	$+1-\varepsilon$	$+1+\varepsilon$	$+\infty$	
f	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	Altogether
f_1	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	three real
f_2	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	roots.
	no roots		no roots		one root		one root		

BRYN MAWR,
January, 1906.

THE MOVEMENT FOR REFORM IN THE TEACHING OF MATHEMATICS IN PRUSSIA.

Ueber eine zeitgemässe Umgestaltung des mathematischen Unterrichts an den höheren Schulen. F. KLEIN. Leipzig, 1904, pp. 82.

Beiträge zur Frage des Unterrichts in Physik und Astronomie an den höheren Schulen. Vorträge gehalten von Behrendsen, Bose, Riecke, Stark, und Schwarzschild. E. RIECKE. Leipzig, 1904, pp. 108.

Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an den höheren Schulen. F. KLEIN und E. RIECKE. Teil I. Leipzig, 1904, pp. 190.

Verhandlungen der Breslauer Naturforscher-Versammlung über den naturwissenschaftlichen und mathematischen Unterricht an den höheren Schulen. Leipzig, 1905, pp. 77.

Bericht der Unterrichts-Kommission der Gesellschaft deutscher Naturforscher und Aerzte über ihre bisherige Tätigkeit. Leipzig, 1905, pp. 57.

THESE works are recent manifestations of an important movement for the improvement of the teaching of mathematics in Prussia, which has been in active progress for quite a number of years.

At the head of the movement stands Professor Klein, of Göttingen, whose views as to the reform needed are expressed in the first work of the above list. This contains five lectures delivered by Klein at a vacation course for teachers held at the University of Göttingen in the spring of 1904. In addition to these there are reprinted several previous publications of Klein's, and a paper by E. Götting on the aim of mathematical instruction in the higher schools (15 pages). The second work contains lectures on topics relative to physics and astronomy delivered at the same course for teachers, and the third consists of the first two bound in one volume.

At the outset, Klein defends himself against two accusations that are usually made against university professors who take active interest in the problems of the schools: first, that they wish to increase the requirements in mathematics; second, that they have in mind exclusively preparation for university work in mathematics. Klein denies emphatically that he takes either of these attitudes. He wishes no raising of the mathematical level in the schools, but rather its horizontal displacement. He is not concerned about what mathematics the prospective student of mathematics in the university may learn in the schools; this student will be taken care of when he gets to the university. It is the non-mathematician that needs special attention — the prospective physician, chemist, jurist. The primary duty of the schools is to give these a mathematical equipment that shall be really fruitful in their subsequent careers.

Klein advocates, not an enlargement of the content of elementary mathematics, but a horizontal displacement of its boundary. Purists define elementary mathematics as the totality of those parts of mathematics in which the concept of limit is not used. This admits much of the theory of numbers, for example, into elementary mathematics, but excludes $\sqrt{2}$ and π. A second definition admits the idea of limit but not in the special form denoted by the symbols dy/dx and $\int ydx$. This would exclude from elementary mathematics the very common ideas of tangent to an arbitrary curve, of its length and area, of velocity, acceleration, etc., and would admit much that is not within the range of the school, for example, an "elementary theory of analytic functions of a complex variable," which, with Weierstrass, avoids Cauchy's theorem and bases all proofs on comparison of infinite power series. A third definition calls everything elementary in geometry which relates to the geometry of Euclid and the ancients. This would admit into elementary geometry the difficult modern researches on the foundations of geometry, and would exclude the simplest modern considerations on variability and transformation of figures; that is, it would keep the spirit of modern geometry entirely out of the schools.

The only definition of elementary mathematics that will be of any use to the school must be a practical one:

In all domains of mathematics those parts are to be called elementary which can be understood by a pupil of average ability without long continued special study.

The content of elementary mathematics, under this definition, is naturally dependent on the time. Improved presentation brings new topics into the elementary field. There can be no question that the determination of the derivative and the integral of x^n (n a positive integer) to-day belongs in the elementary field, and Klein's chief thesis is:

The introductory development of the function concept, the first introduction to analytic geometry, the beginning of the differential and integral calculus, should be a part of the course of every pupil.

The explanation and discussion of this thesis constitutes the backbone of the series of addresses. The early and continued use of graphic representation of simple functions is urged; for example of

$$y = ax + b, \quad y = x^2, \quad y = 1/x$$

in "Untersekunda" (age 14–15), together with varied applications, especially to the mathematical discussion of physical phenomena. As a minimum, the law of falling bodies, the motion of the pendulum and wave motion should be understood mathematically. The treatment must be sufficiently formal to assure real understanding, but the main end is clear grasp of the fundamental concepts and their concrete significance.

"It is often said that the beginning of higher mathematics (*i. e.*, of the differential and integral calculus) means a revolution in the thinking of the student. The plan of instruction which I am urging is meant to spare the student this revolution by accustoming him from the beginning to the processes of thought which are later to prove fruitful. This 'revolution' is merely a test of the unsuitability of the earlier work to the later aims."

It would be interesting to cite many other instructive details of these addresses. In fact the whole work is well worth translating into English, but what has been said above must suffice for the present writing, with the remark in conclusion that here, as elsewhere, Klein also takes occasion to lay stress on the demand that the universities give special attention to the needs of prospective teachers of mathematics. It will be recalled that three years' university study of mathematics are a part of the professional preparation of every teacher of secondary mathematics in Prussia. Consequently, it is likely that prospective teachers constitute the majority of the mathematical students at the universities, and yet until recently the latter have given no explicit attention to their needs. The subject matter and the methodology of elementary mathematics deserve special attention from the university, and Klein announces for the winter next following a course by himself for prospective teachers under the title: "Concerning mathematical instruction in the higher schools."

The fourth and fifth reports relate to activities of the German society of natural scientists and physicians, dating from the year 1901, when the so-called "Hamburg theses" relative to instruction in the biological sciences were presented. In the consideration of these theses at the session at Cassel in 1903, it was decided, upon motion of Professor Klein, to make the entire field of instruction in the natural sciences and mathematics the subject of thorough discussion. This discussion took place at Breslau in 1904 and No. 4 of the above list is the published report of this discussion. Among the formal papers presented were one by Fricke on "The present status of instruction in the natural sciences and mathematics in the higher schools" and one by Klein under the title "Remarks on instruction in mathematics and physics." Here also Klein urged the importance of the function concept and of the elements of the calculus in the mathematical side of modern culture, the need of considering mathematics in closer relation with its applications in the exact sciences and in all phases of life capable of precise formulation, and the danger of allowing the formal logical side to predominate too strongly in instruction in mathematics. Both speakers lay marked stress on the importance of thorough preparation of teachers, and emphasize the duty of the universities in this connection, not only in the matter of scientific equipment, but also in that of methodic and didactic preparation. Klein also advocates that teachers be given regularly a semester's leave of absence to give them opportunity to survey the progress of their science, and by personal relations and inspection on travels here and there, to make acquaintance with the progress that is being made in this field, and especially to see what part their own subject is playing in the culture of our day.

At the close of this discussion, a commission of twelve was appointed, including Klein and other mathematicians, to draw up courses of study in mathematics and the natural sciences, bringing into unified and practical form the various propositions made in the discussion. The report of this commission is the fifth work named above, and an educational document of great significance.

After a general outline by the chairman, Professor A. Gutzmer, of the University of Jena, of the work of the commission, its leading recommendations and the principles which governed it in making them, there follow detailed curricula in each of the subjects mathematics, physics, chemistry, botany, and zoology. Only that in mathematics concerns us here. Not undervaluing logical training, the commission regarded strengthening of space intuition and training to the habit of functional thinking as the most important task of instruction in mathematics. As the goal to be reached by this instruction there were set up a scientific survey of the organization of the whole field covered in the school, a certain facility of regarding things mathematically, and above all insight into the importance of mathematics in the exact knowledge of nature, and in the culture of the day in general.

In accordance with this general point of view, the detailed curriculum gives new prominence to the concrete phases of mathematics (drawing, graphic methods), to the function concept and the representation of functions by curves, and to the applications of mathematics. The introduction of the elements of the calculus was left to the option of the teacher.

Though relating directly only to German conditions, these publications are of significance beyond the borders of that country. The first and the last will prove of most interest to the general reader. The movement itself, of which all these works are the outgrowths, is undoubtedly of the first importance. In addition to active interest on the part of leading mathematicians and of the strong and influential society appointing and supporting the commission whose report we have considered, other evidences of a widespread and significant participation in the movement are not lacking. Of these only the representation of the Deutsche Mathematiker-Vereinigung on the Commission, and the space devoted to pedagogic matters in its *Jahresbericht*, need be mentioned here. While this movement is directly concerned only with mathematics in Germany, it is nevertheless of international significance, both on account of the fundamental nature of the changes which it proposes and the weight attaching to the names of the societies and men that are engaged in it. In its essentials, it is in close harmony with movements of the day for improvement of the teaching of mathematics in England, in France and in the United States, which agree in demanding:

1. That the presentation take more careful account of the pupils' powers; that it be psychological as well as logical; that consequently the intuitional, the inductive, the concrete, be given prominent place in the teaching of mathematics.

2. That the amount of complex or merely technical work be diminished.

3. That, in addition to its disciplinary value, the importance of mathematics for a clear understanding of nature, and the fundamental character of mathematics in the fabric of modern civilization be brought effectively before the pupils.

J. W. A. Young.

The University of Chicago,
January 24, 1906.

VECTOR ANALYSIS.

Vorlesungen über die Vektorenrechnung. Von E. Jahnke. Leipzig, B. G. Teubner, 1905. xii + 235 pp.

The entering class at an American college is more or less prepared on elementary geometry and algebra, and perhaps on trigonometry. The problem of collegiate instruction has, therefore, one fairly well-defined premise. All the students who continue their work in mathematics, for no matter what purpose, must study analytic geometry and calculus. From this