HOW SHOULD THE COLLEGE TEACH ANALYTIC GEOMETRY? *

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IN most American colleges, analytic geometry is an elective study. This fact, and the underlying causes of this fact, explain the lack of uniformity in content or method of teaching this subject in different institutions. Of many widely divergent types of courses offered, a few are likely to survive, each because it is best fitted for some one purpose. It is here my purpose to advocate for the college of liberal arts a course that shall draw more largely from projective geometry than do most of our recent college text-books. This involves a restriction of the tendency, now quite prevalent, to devote much attention at the outset to miscellaneous graphs of purely statistical nature or of physical significance. As to the subject matter used in a first semester, one may formulate the question : Shall we teach in one semester a few facts about a wide variety of curves, or a wide variety of propositions about conics?

Of these alternatives the latter is to be preferred, for the educational value of a subject is found less in its extension than in its intension; less in the multiplicity of its parts than in their unification through a few fundamental or climactic principles. Some teachers advocate the study of a wider variety of curves, either for the sake of correlation with physical sciences, or in order to emphasize what they term the analytic method. As to the first, there is a very evident danger of dissipating the student's energy; while to the second it may be replied that there is no one general method which can be taught, but many particular and ingenious methods for special problems, whose resemblances may be understood only when the problems have been mastered. No one can lay down rules *a priori* which would enable the student to rediscover the theorems now known concerning conics — if these were once lost. Were there such a single well-defined method or art of creating geometry, should we still be waiting for the completion of the theory of plane quartics, or for even an outline of the theory of quintics? It is rather the art of asking questions

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* Read before the Association of Teachers of Mathematics of the Middle States and Maryland, at Annapolis, Md., December 1, 1905.

that is to be taught. How can one learn what kind of questions will prove fruitful, save by examining those which have been asked and answered by geometricians in the past?

If the hundreds of important theorems in conics were really unrelated, the argument for their study simply as pertaining to conics would be weak. But just there lies the point; modern geometry has found general principles which unify the whole subject. Every part is connected with every other part. The student learns to employ many methods of inquiry or of proof, but does not lose himself in endless diversity. And while no one can affirm that a branch of science is complete, yet all historians point to the theory of conics as a model both in the large number of its particular propositions and in the possession of general concepts and theorems. It is the most optimistic dream of workers in other branches of mathematics that their favorite special subject may some day reach a stage of development equally advanced with that to which two thousand years have brought this ancient favorite of geometricians.

Conics is an ample range for one semester, and needs but little enlargement for a second. Of the methods to be employed not much need be said to experienced teachers. For the first two weeks, constant drill work on points, distances, areas, circles, and parabolas. By that time the notion of a language for definite use will impress itself. The desire to express a distance in rectangular coördinates calls up a radical covering the sum of two squares, and conversely the sum of two squares of differences of like coördinates recalls involuntarily the motion of a distance; the equation of a circle comes to look to the eye like the definition of a circle in words. A parabola is thought of when an equation separates into one linear part and one square of a different linear expression. Proportional division becomes as familiar as the inflection of a Latin noun, and the area of a triangle in terms of vertices, a string of six terms, comes to look like a moderately long German word. Analysis and synthesis have begun to be instinctive, as they are when one has taken the first steps in a foreign language.

Next comes the patient extension of the vocabulary simultaneously with the study of new geometrical notions. Slope, intercepts, systems of lines, pencils of conics prove easy one after another, and the first milestone is reached when after twenty hours of class-room work the student perceives that every linear expression denotes a point-function with niveau lines straight and parallel, and that all such expressions may serve as coördinates equally well with his favorite x and y.

This is the point from which geometry should be the main object, and the properties of conics should be developed. Of course metrical properties must come first. The parabola with its axis, focus, vertex, and directrix must be fixed by drawing practice; the bisection of the sub-tangent by the vertex will naturally be the first theorem, and the constancy of the sub-normal the next. Mutually perpendicular tangents meeting on the directrix, and circles circumscribed to circumscribed triangles are important enough to demand attention from even the most hurried student. The diameters of parabolas and of all other conics should be studied together, and conjugate pairs of diameters with their use as axes of reference.

So far, probably every teacher will agree, the student's attention should be confined to metric properties; and indeed in view of the number, the beauty, the practical value of metric properties of conics, it is not surprising that most American text-books include nothing more general. But it is to be remembered that the aspect of this large body of truths to the beginner is not the same as to the more advanced student or teacher. The former is first pleased with the novelty, then wearied with the multiplicity of disconnected facts, and will soon become bewildered and discouraged. A passage from Taylor's Ancient and modern geometry of conics puts the case so well that I will quote it here. "The mind of the tyro is commonly overwhelmed with a multitude of details not reduced to any system, demonstrations are put before him in an unsuggestive form which gives no play to his inventive faculty; and thus it comes to pass that of the many students so few turn out genuine geometers. Let the learner be furnished with principles, and not alone with fully explained facts, and be continually stimulated to exertion by the intense pleasure of finding something left to discover for himself." (P. LXXVII, § 4.) In contrast to this attitude of the neophyte confronted with the wealth of existing theorems of metric character, let any teacher, familiar with projective geometry, review Pascal's chapter (Repertorium der höheren Mathematik, part II) or any other brief synopsis of this subject. He will note the rising tide of pleasure and admiration as well-known and less known discoveries succeed one another, in all some sixty or seventy facts of major importance; and will probably observe that half of his satisfaction is referable to his serene conviction that all these things flow as easy deductions from a very small number of projective theorems.

Projective geometry is acknowledged to furnish a fascinating continuation or climax to metric geometry; but if it can also serve as a clue, a map amid the intricacies of the elementary theories, why should it not be employed for that purpose? After working two-thirds of a semester to acquire an understanding of the objects and a preliminary knowledge of the theorems of metric cônics, let the student learn the few fundamental and comprehensive projective theorems. First of all comes, of course, the relation of conjugate points, commonly stated in the form: If a point P lies on the polar of a point Q, then conversely Q will lie on the polar of P. Without much assistance the student can see that this theorem contains no reference to axes of coordinates, and will suspect that it points to a more général geometry than he has yet studied. Next he can be made acquainted with the harmonic position of four points, or with the anharmonic ratio of any four collinear points or copunctual lines, and then return for practice to the theorems on conjugate diameters. Then he should learn that the anharmonic aspect of four points on a conic is the same from all points of that conic, and the dual theorem on tangents. He has now in hand the projective (organic) generation of the general conic, the unifier of many facts previously independent.

Returning to the metric basis, I would then establish the use of trilinear coordinates by the use of the formula for distance of point from line. Thereupon one may present Salmon's proofs for Carnot's theorem concerning the segments cut by a conic on the sides of a triangle; and if the theorem of Menelaus on the straight transversal of a triangle has been introduced earlier, Pascal's hexagon may be made the terminus of the first semester's study. Less than this ought not to satisfy us. As no college student of chemistry ought to be a whole semester in ignorance of Mendeléeff's table; as no college student of physics should pass an examination before learning, however crudely, some reasons for believing in the conservation of energy; so in analytic geometry let us give even the beginner in his first course some knowledge of Pascal's immortal theorem.

In the college course, the average student of sciences cannot afford time for more than one semester of geometry, even though the analytic auxiliaries may seem to open a whole new world to his vision. He must hasten to master the elements of the differential and integral calculus and attack applied mathematics. Hence the necessity of making the first semester of analytic geometry less slow and richer in material than is at present customary. But provision ought to be made for another year's work for those who wish to become mathematicians. Their second semester should make them acquainted first with quadric surfaces, both as individual surfaces and as associated in a pencil and in a confocal system, but next and by no means less in importance, with the remaining chief propositions concerning plane conics. These are perhaps those of Desargues and Steiner on two perspective triangles, that of Brianchon on the sextette of tangents, with modifications for quintettes, quartettes, and triads; Hesse's famous theorem on the six vertices of a complete quadrilateral, four of which form two conjugate pairs in a polarity, and the theorems of Poncelet concerning polygonal lines inscribed to one conic and circumscribed to another. And lack of famous names should not cause us to omit those relating to pairs of triangles, the first where the triangles are polar reciprocals with respect to a conic, and the second where each is self-polar.

These latter theorems, with due attention to the powerful methods of projection and reciprocation, may well enough fill up the second semester; of course each one should be the occasion of two or three careful exercises in drawing and in schematic constructions. The third semester will find its most appropriate material in mutually apolar systems of conics, and in a rapid discussion of plane cubic curves. At present this is, so to speak, the end of improved roads in one direction, and students who come so far may easily obtain such outlooks as the ambitious are sure to desire; and they will have sufficient experience of travel to warrant them in undertaking explorations upon their own account. Apropos of this we may notice Loria's summary of the present situation, in his valuable compend of Special plane curves (pages 219–20).

"The theory of curves of the third order has clearly become, so to speak, a province of the realm of mathematics; the theory of curves of the fourth order is a territory only partly subjected to this kingdom; but the theory of curves of any particular order higher than the fourth still asserts its independence and resists stubbornly every attempt to introduce law and order. If we except [certain special researches upon quintic curves] we may say that up to the present nothing has occurred to illuminate the thick darkness which conceals from us the properties of those curves which should come next after conics, cubics and quartics. Not only so, but there is not even any special curve of fifth order which has received a particular name."

The course here advocated is somewhat modernized from the traditional type of college courses in geometry, but it is not, therefore, chimerical. A survey of the field should convince us that in three semesters a college student can become posted on the essentials of conics, quadric surfaces, and cubic curves. Individual teachers will prefer a different order from that here advocated, either for theoretical reasons or on account of the special qualifications of the students under their charge; but it is believed that the main thesis will command general approval — the most modern methods and the most general propositions of present day geometry must be made available for the college student.

VASSAR COLLEGE,
November 29, 1905.

FOUR BOOKS ON THE CALCULUS.

Die Anfangsgründe der Differentialrechnung und Integralrechnung. Für Schüler von höheren Lehranstalten und Fachschulen sowie zum Selbstunterricht. Dargestellt von DR. RICHARD SCHRÖDER. With numerous exercises and 27 figures in the text. Leipzig, B. G. Teubner, 1905. vii + 131 pp.

Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen. Zusammengestellt von DR. ROBERT FRICKE. Fourth edition, with 74 figures in the text. Braunschweig, Friedrich Vieweg und Sohn. xv + 217 pp.

Repetitorium und Aufgabensammlung. Von DR. FRIEDRICH JUNKER. Second, improved edition, with 46 figures in the text. Leipzig, G. J. Göschen, 1905. 16mo. 127 pp.