los "* are here designated as "improper" continued fractions and their value determined by the following definition: The expression

$$(1) \qquad b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$$

determines the quantities Z_k by the equations

$$(2) \qquad \begin{gathered} Z_n = b_n, \quad Z_{n+1} = 1, \\ Z_m = b_m Z_{m+1} + a_{m+1} Z_{m+2} \\ (m = n-1, n-2, \cdots, 1, 0). \end{gathered}$$

The value of (1) is Z_0/Z_1. The only fractions which now fail to have values are those for which Z_1 is zero. If $Z_\kappa = 0$ for any value of κ other than 1, the fraction is improper. This definition leads to no new extension of the notion of an infinite continued fraction because improper convergents had been used previously even where the corresponding continued fractions were regarded as meaningless. Among the modern developments added to the chapter on infinite continued fractions, it is interesting to note the theorems of Van Vleck published in the second volume of the *Transactions*.

To sum up this notice and also the review of the first volume in last December's BULLETIN — the work as a whole gives an effect of conservatism, maturity and poise. It is not as likely as the modern French books to stimulate research, but it has a permanent value as a repository of accurate information about the conventional functions and processes of analysis.

OSWALD VEBLEN.

Lezioni sul Calcolo degli Infinitesimi, date nella R. Università di Modena da ETTORE BORTOLOTTI. Raccolte dal Dr. ARMANDO BARBIERI. Modena, 1905. vii + 61 pp.

BORTOLOTTI'S Calcolo degli infinitesimi is a short course of lectures on various questions that arise in connection with the determination of the relative orders of two infinitesimals (infinites)

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*Because $2 - \frac{1}{1} - \frac{1}{2} = 0$. The significance of this definition of "improper" continued fractions may perhaps be suggested by the equation

$$3 + \frac{2}{7} + \frac{5}{0} = 3 + \frac{2 \cdot 0}{0 \cdot 7 + 5} = 3.$$

$f(x)$ and $\phi(x)$, and consequently includes a discussion of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, etc. The book is intended for comparatively young students. It begins by defining the upper and lower bounds of indetermination of $\psi(x)$ as x approaches a limiting value and recapitulating some general theorems on limits. It then defines the equality of orders of $f(x)$ and $\phi(x)$ as meaning that $|f(x)/\phi(x)|$ has finite and positive bounds of indetermination. If the limit of $f(x)/\phi(x)$ exists and is infinite as x approaches a limiting value, $f(x)$ is said to be an infinitesimal of lower order than $\phi(x)$; $f(x)$ is of order m if of the same order as $(x-a)^m$.

The difficulties of this definition, for example that functions exist whose order is less than $m+\alpha$ and greater than $m-\alpha$ for every $\alpha > 0$ without being equal to m, are exhibited and used to lead up to Cauchy's definition: $f(x)$ is infinitesimal of the same order as $\phi(x)$ if for every constant $\alpha > 0$,

$$\lim_{x=a} \frac{f(x)}{[\phi(x)]^{1+\alpha}} = \infty, \lim_{x=a} \frac{f(x)}{[\phi(x)]^{1-\alpha}} = 0.$$

This, as Bortolotti points out, is equivalent to saying that when $\epsilon(x)$ is defined by the equation $f(x)/\phi(x) = [\phi(x)]^{\epsilon(x)}$,

$$\lim_{x=a} \epsilon(x) = 0.$$

This definition is subject to the inconvenience of requiring two functions to be of the same order in some cases where their ratio approaches the limit ∞, but it has the advantage of extending the notion of order to a wider class of functions than is reached by the first definition.

After developing the properties of Cauchy's definition of order, Bortolotti goes on to expound in a clear and rigorous way the rules derived by l'Hôpital, Cauchy and Stolz for the evaluation of the indeterminate forms $0/0$ and ∞/∞. Here, it is to be remarked that to the hypothesis of the theorem on page 25 should be added the sentence, "If $\phi(x)$ is infinitesimal, so is also $f(x)$."

These more or less familiar theorems are followed by methods of determining the relative orders of functions by examining the double ratios

$$\frac{f'(x)}{\phi'(x)} : \frac{f(x)}{\phi(x)}, \quad \frac{\Delta f}{\Delta \phi} : \frac{f}{\phi}.$$

Some of these criteria might perhaps have been stated to better advantage in terms of the derivatives of the logarithms of $f(x)$ and $\phi(x)$. The book ends with a consideration of the indeterminate forms $\infty - \infty$, 1^∞, etc.

Taken as a whole, the book is not only clear, readable, and fairly thorough, but has a good set of references to other works and a rich list of problems which may be useful in elementary calculus courses. It therefore seems to deserve a place in any fairly complete mathematical library.

OSWALD VEBLEN.

Abstrakte Geometrie. By KARL THEODOR VAHLEN. Leipzig, B. G. Teubner, 1905. xi + 302 pp.

A book following the general outline of Vahlen's Abstrakte Geometrie would be very useful for giving a general view of the recent studies on foundations of mathematics. This is sufficiently indicated by the titles of the chapters: I, Foundations of arithmetic; II, Projective geometry (theorems of connection); III, Projective geometry (theorems of order); IV, Affine geometry (euclidean and non-euclidean); V, Metric geometry.

Unfortunately, however, the book is not characterized by that precision of language which is indispensable in any discussion of such a subject. The reader is constantly confronted with statements which are incorrect if taken literally and which, if not taken literally, are open to more than one interpretation. Many of the author's postulates are labeled by him as definitions. Moreover, there are places where it is very difficult to determine which of the previously stated hypotheses are being used and which are not. As a consequence, the reviewer is able to state hardly a single new *result* which is surely established by this book.

On the other hand, there are many *suggestions* of methods which if rigorously carried out would probably lead to interesting and elegant results. For example, the notion of planar order is defined not by means of coördinates as in the usual analysis, nor by the way in which a straight line intersects a triangle (according to Pasch) but by means of postulates in terms of the right and left sides of a line with respect to a given sense on the line. One is thus enabled to deal at once with the most general type of planar-ordered set without presupposing anything about a plane in which it lies.