$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\right]^{(2)}+\sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2},$$

where the exponent in parenthesis signifies that the expression to which it is attached is to be squared, and after squaring

$$\left(\frac{qy}{qu}\right)^2,\quad \left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right),\quad \left(\frac{qy}{qv}\right)^2$$

are to be replaced by

$$\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right),\quad \left(\frac{qy}{qv}\right)\left(\frac{q^2y}{quqv}\right),\quad \left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right),$$

respectively.

W. A. MANNING,
Secretary of the Section.

PROJECTIVE DIFFERENTIAL GEOMETRY.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5–8, 1906.

BY PROFESSOR E. J. WILCZYNSKI.

THESE four lectures were devoted to an exposition of the principal results belonging to the subject of projective differential geometry. The place of this subject in a systematic treatment of geometry is indicated by the following discussion.

A first important basis for the classification of the various geometries is furnished by the group concept. There is metric geometry, projective geometry, the geometry of the birational transformations, to mention only the most important. Together with this classification by means of the characteristic groups, there is the distinction between differential and integral geometry. The differential properties of a geometric configuration merely depend upon the fact that in a certain, perhaps very small, region, certain conditions of continuity are satisfied, that derivatives of a certain order exist, etc. These differential properties are studied by means of the differential calculus and do not depend upon the nature of the configuration (curve, surface, etc.) as a whole, but merely upon its character in a certain neighborhood. Opposed to the differential, we have the integral properties which depend upon the configuration as a whole. Thus within each geometry, characterized by its group we may still distinguish two divisions; we have differential and integral metric, differential and integral projective geometry; the geometry of birational transformations is likewise composed of two such divisions. Integral projective and differential metric geometry have been systematically developed. It is noticeable that the same is not true of differential projective or of integral metric geometry. In the nature of things differential geometry is more easily accessible than integral geometry. For the relation between the two is precisely that of the differential to the integral calculus. From the properties of differential geometry those of integral geometry may be obtained by integration, a process which requires the invention of special methods for every particular case. If, therefore, we can speak of an integral projective geometry as existing, it is nevertheless only in a very special sense. In fact, the configurations of projective geometry have, in almost all cases, been assumed to be algebraic, a restriction which is equivalent to an a priori integration. Leaving the algebraic cases aside, it is clear, therefore, that integral geometry, taken in the general sense, must be preceded by differential geometry. The metric half of this latter subject has been occupying the attention of mathematicians since the days of Monge and Gauss, and in the hands of their successors has reached a high degree of perfection. The same cannot be said of projective differential geometry. These lectures give an account of the first systematic treatment of a part of this subject, namely the projective differential geometry of curves and ruled surfaces.

The theory of curves in a space of $n-1$ dimensions is based upon the theory of the invariants and covariants of a linear homogeneous differential equation of the nth order, a point of view due to Halphen and developed by him in an admirable series of papers. The theory of ruled surfaces is based upon the consideration of a system of linear differential equations of the form

$$(A) \qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0. \end{aligned}$$

In connection with this system, consider the infinite group G of transformations

$$(G) \qquad \eta = \alpha(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \xi = f(x),$$

where α, β, γ, δ, f are arbitrary functions of x. The transformations of G convert (A) into another system of the same kind. The problem presents itself: to find the invariants and covariants of the system (A) under the transformations of the group G. This problem has been completely solved.

Let (y_i, z_i), $(i = 1, 2, 3, 4)$ be four pairs of simultaneous solutions of system (A). Then it may be shown that the most general pair of solutions is given by the equations

$$y = \sum_{i=1}^{4} c_i y_i, \quad z = \sum_{i=1}^{4} c_i z_i,$$

involving four arbitrary constants $c_1, \cdots c_4$, provided that the determinant

$$D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

is not identically equal to zero. Four such pairs of functions, whose determinant does not vanish, and which satisfy system (A), constitute a fundamental system of simultaneous solutions. Conversely any four pairs of this kind determine a system of form (A) of which they constitute a fundamental system.

Interpret $(y_1, \cdots, y_4)$ and $(z_1, \cdots, z_4)$ as the homogeneous coordinates of two points p_y and p_z. As the independent variable x changes, p_y and p_z describe two curves C_y and C_z, the integral curves of system (A). There is a definite correspondence between the points of these curves, those being corresponding points which belong to the same value of x. Join corresponding points p_y and p_z by a straight line h_{yz}. The ruled surface S which is the locus of these lines is called the integrating ruled surface of system (A). It is easy to show that any non-developable ruled surface may be defined by a system of form (A); the developables are excluded by the condition that the determinant D shall not be equal to zero.

The transformations of the group G leave the ruled surface S invariant. In fact, their geometric significance is merely to replace the two curves C_y and C_z upon S by two other curves C_η and C_ζ upon S, the point-to-point correspondence being again determined by the generators of S. The transformations of G further involve the most general change of the independent variable. An invariant of system (A) under the transformations of the group G therefore has a significance for the ruled surface S which is independent of the particular curves upon S chosen as fundamental curves, and of the choice of the independent variable. Moreover this significance is of a projective nature, since any projective transformation of the surface S gives rise to the same system of form (A), and since the most general integrating ruled surface of (A) is a projective transformation of any particular one.

Upon these considerations the lecturer based his theory of ruled surfaces which, together with some known results, contains a great variety of new ones. It is impossible to give an intelligible account of the details of this theory without greatly exceeding the space at our disposal. It is expounded in detail in the lecturer's recent treatise.* This brief abstract will suffice to indicate the character of the subject under investigation and the nature of the methods employed.

ON LOCI THE COORDINATES OF WHOSE POINTS ARE ABELIAN FUNCTIONS OF THREE PARAMETERS.

BY PROFESSOR J. I. HUTCHINSON.

(Read before the American Mathematical Society, September 3, 1906.)

A particular case of a surface the homogeneous coordinates of whose points are theta functions of three variables u_1, u_2, u_3 (connected by a relation $\vartheta = 0$) is given by Humbert † and studied in considerable detail by means of this parametric rep-

---

* E. J. Wilczynski. Projective differential geometry of curves and ruled surfaces. Leipzig, B. G. Teubner, 1906.

† "Sur une surface du sixième ordre liée aux fonctions abéliennes de genre trois," *Liouville*, 1896, pp. 263–293.