$\cdots$, ϑ_5 respectively. Then $x_i = \vartheta_i^2 (i = 0, 1, \cdots, 4)$ determine a spread S' of degree 12 having 61 quadruple points. As in the preceding example, to each of the half-periods α, β, γ corresponds a two-spread (a two-dimensional locus) of degree 4.

Since two functions ϑ_i vanish simultaneously for 12 half-periods, three of which are α, β, γ, it follows that the flats $ax_i + bx_j = 0$ intersect S' in 9 of its nodes. If we require it to pass through a tenth not lying in either x_i or x_j, there are 29 points to choose from, and hence we determine $29.15 = 435$ surfaces of degree 12, each having 10 quadruple points.

Every flat of the form $ax_i + bx_j + cx_k = 0$ contains 3 nodes of S' and can be made to pass through two other nodes by properly choosing the constants. There are thus determined $20.45 = 900$ surfaces of degree 12 which have 5 quadruple points.

The coördinate flats $x_i = 0$ contain doubly covered sextics. Consider, for example, the sextic $\vartheta_0 = 0$, which we denote by T'. It has 25 nodes, 5 cubic curves $\vartheta_i = 0 (i = 1, \cdots, 5)$ lying in singular tangent planes, 18 quartic (genus zero), 30 quintic, and 10 sextic twisted curves.

Cornell University,
August, 1906.

ASSOCIATED CONFIGURATIONS OF THE CAYLEY-VERONESE CLASS.

BY DR W. B. CARVER.

(Read before the American Mathematical Society, September 3, 1906.)

In this paper, S_n will be used to denote a flat space of n dimensions; and the notation $C^{\nu}_{n,r}$ and the words "chiastic," "copoint," etc., will be used in the sense defined in the author's earlier paper * on these configurations.

Given five planes in S_3, we can, in general, construct the polar point P of any one of the five with respect to the 4-point determined by the other four planes. The five points so constructed, when joined by lines and planes, give a complete 5-point chi-

* "On the Cayley-Veronese class of configurations." *Transactions Amer. Math. Society*, vol. 6, pp. 534–545 (October, 1905).

astic with the complete 5-plane; and the 5-point and 5-plane are said to be *associated*.*

Projecting this entire figure upon a plane (by what Veronese † calls an *univoca*, and his German translators an *eindeutige* projection) we have the following:

A $C^3_{5,2}$ (the Desargues configuration of two perspective triangles with their center and axis of perspective) breaks up in five ways into a $C^3_{4,2}$ (a complete 4-point) and a $C^2_{4,2}$ (a complete 4-line). Each $C^2_{4,2}$ and the corresponding $C^3_{4,2}$ determine a point P, which may be called the polar point of the $C^2_{4,2}$ with respect to the $C^3_{4,2}$; and the five points so determined, when joined by lines, form a $C^4_{5,2}$ (a complete 5-point) chiastic with the $C^3_{5,2}$. This $C^3_{5,2}$ may be said to be associated with the $C^4_{5,2}$.

This simple conception may be doubly extended, and we obtain the following general theorem:

Let there be given, in S_r, a $C^n_{n+2,r}$. This $C^n_{n+2,r}$ breaks up in $n+2$ ways into a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$, and we can construct the polar point of each $C^{n-1}_{n+1,r}$ with respect to the corresponding $C^n_{n+1,r}$. The $n+2$ points so constructed give a $C^{n+1}_{n+2,r}$ chiastic with the $C^n_{n+2,r}$, and we may speak of the two as being associated.

It remains now to give a method for constructing the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^n_{n+1,r}$. This can be done, without going outside of the space S_r in which the configurations lie, in the following manner:

Let the $C^n_{n+1,r}$ be lettered with $n+1$ letters, as described in the author's earlier paper. Any $n-1$ letters denote a line of the $C^n_{n+1,r}$ and on this line lies a point of the $C^{n-1}_{n+1,r}$. The point of the $C^{n-1}_{n+1,r}$ has a polar point P_{n-1} with respect to the two points of the $C^n_{n+1,r}$ lying on that line. (Let this P_{n-1} have a subscript made of the $n-1$ letters belonging to the line.) Any $n-2$ letters denote a triangle, or $C^2_{3,r}$, of the $C^n_{n+1,r}$. If we join the P_{n-1} on each line of such a $C^2_{3,r}$ to that point whose symbol contains the $n-2$ letters of the $C^2_{3,r}$ but not the remaining letter of the P_{n-1}, the three such joins meet in a point P_{n-2}. In the $C^n_{n+1,r}$ $n-3$ letters denote a $C^3_{4,r}$. If we join the P_{n-2} of each $C^2_{3,r}$ in such a $C^3_{4,r}$ to the proper point (de-

---

* Cf. Morley, "Projective coordinates," *Transactions Amer. Math. Society*, vol. 4. See p. 292.

† Cf. Grundzüge der Geometrie, p. 614.

termined as before), these four joins meet in a point P_{n-3}. * * * Finally, if we join each P_1 to that point of the $C^n_{n+1,r}$ whose symbol does not contain the letter of the P_1, these $n+1$ joins meet in a point P which is the polar point of the $C^{n-1}_{n-1,r}$ with respect to the $C^n_{n+1,r}$.

The dual theorem for S_r gives a polar copoint, or S_{r-1}, of a $C^{r+1}_{n+1,r}$ with respect to a $C^r_{n+1,r}$; and, when $r=2$, we have, in the plane, a polar line of a $C^3_{n+1,2}$ with respect to a $C^2_{n+1,2}$. This special case in the plane is treated by Caporali, but his entire method is different from the method used in this paper.

If, having constructed in S_3 the polar point P of a plane with respect to the 4-point determined by four other planes, we project the whole figure upon a plane, *using P as the center of projection*, we obtain a $C^3_{4,2}$ and a $C^2_{4,2}$ which are associated, and the polar point of the former with respect to the latter is evidently indeterminate. In general, suppose we take, in S_n, the polar point P of a copoint, or S_{n-1}, with respect to the $(n+1)$-point determined by $n+1$ other copoints, and then project this figure upon an $S_r (r < n)$ from a co-S_r, or S_{n-r+1}, taking an S_{n-r+1} incident with the point P. We will thus obtain in S_r a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ which are associated; and the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^n_{n+1,r}$ is indeterminate. *Whenever* a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ are associated, the polar point of the latter with respect to the former is indeterminate. This fact also Caporali notes for the plane case.

This polar idea for configurations admits of one more extension. If we have, in S_r, a $C^{n-k}_{n+1,r}$ chiastic with a $C^n_{n+1,r}$ $(k \leqq r-1)$,* we may define the polar S_{k-1} of the $C^{n-k}_{n+1,r}$ with respect to the $C^n_{n+1,r}$ thus:

There are ∞^{k-1} $C^{n-1}_{n+1,r}$'s chiastic † with the $C^n_{n+1,r}$ and also with the $C^{n-k}_{n+1,r}$, and any k of these $C^{n-1}_{n+1,v}$'s are sufficient ‡ to determine the $C^{n-k}_{n+1,r}$. Construct the polar point of each of k of these $C^{n-1}_{n+1,r}$'s with respect to the $C^n_{n+1,r}$ and these k polar points will lie upon an S_{k-1} which is the polar S_{k-1} of the $C^{n-k}_{n+1,r}$ with respect to the $C^n_{n+1,r}$.

There is of course a dual theorem which defines a polar S_{r-k} of a $C^{r+k}_{n+1,r}$ with respect to a $C^r_{n+1,r}$.

---

* Author's paper, *loc. cit.*, paragraph under theorem II.

† *Loc. cit.*, theorem VI.

‡ *Loc. cit.*, theorem V.