will be the vertices of a polygon with respect to which and $ABCDEF$ . . . there is an indefinite number of in- and circumscribed polygons.

If $ABCDEF$ . . . be any *regular* polygon of an even number of sides and if AC intersect FB and BD in a, b respectively, CE intersect BD and DF in c, d respectively, etc., then by the properties of the regular polygon there will be closure for the midpoint of AB (and so for all points of AB) if AB be projected on BC, BC on CD, CD on DE, etc., from the poles a, b, c, etc., respectively. Also there will be closure for the midpoint of AB and so for all points of AB if AB be projected on BC, BC on CD, etc., from the midpoints of AC, BD, CE, etc., respectively. If $ABC \cdots$ have an *odd* number of sides and the projections be made from poles corresponding to those indicated in these two theorems, there will be closure for all points if the projection be made twice round the polygon.

If $ABCDE$ be any pentagon and if the points (AC, BD), (BD, CE), (DA, EC), (BE, DA), (AC, EB) be named e, a, b, c, d respectively, then if $ABCDE$ and $abcde$ be regarded as two simple pentagons there will be a poristic system of pentagons if ab be projected on bc, bc on cd, cd on de, de on ea, ea on ab from the vertices B, C, D, E, A respectively. This may be proved by testing for closure when the vertices of $ABCDE$ or of $abcde$ are the points projected.

[Professor Morley has pointed out to me that the simple pentagon $ACEBD$ and any one of the variable pentagons constitute the ten-point, ten-line configuration of Desargues's two perspective triangles.]

VASSAR COLLEGE,
October, 1906.

HERMITE'S WORKS.

Oeuvres de Charles Hermite, publiées sous les auspices de l'Académie des Sciences par EMILE PICARD. Vol. I. Paris, Gauthier-Villars, 1905. 8vo. xl + 498 pp.

I.

ON January 14, 1901, Charles Hermite passed away. A contemporary and zealous disciple of Gauss, Jacobi, and Dirichlet, the friend and generous rival of Cayley, Sylvester, and Brioschi, his death snapped one of the few remaining links connecting the present with the heroic days of the last century.

Hermite was born December 24, 1822, at Dieuze, Lorraine. His parents being in comfortable circumstances, he was early sent to continue his studies at Paris. Here at the lycée Louis le Grand, he had the good fortune to come under the charge of Professor Richard who, as may be remembered, had taught Galois in the same class of mathématiques spéciales some twelve years before.* Hermite, like Galois, early manifested an extraordinary talent for mathematics. Neglecting the regular courses of study, he read with greatest ardor the masterpieces of Euler, Lagrange, Gauss, and Jacobi; and such was the precocity of his genius that while still at Louis le Grand he published two papers and had others well under way.

In 1842, Hermite entered the Ecole Polytechnique as sixty-eighth in his class. His love for the higher mathematics had left him little leisure to prepare for examinations; hence his poor standing. His stay at this famous school was, however, destined to be short. From birth Hermite had suffered from an infirmity of the right leg and had to use a cane. On this account it was now decided by the authorities that he should not be eligible to any of the government positions which are given to the graduates of the Ecole. Hermite, therefore, left at the end of the first year. Although brief, his attendance at the Ecole Polytechnique was not without effect on his further career. For, while here, he was encouraged by Liouville to compose the first of those remarkable letters to Jacobi (January, 1843) in which the genius of Hermite shone with such extraordinary luster.

A rapid and brilliant academic career would have been the natural reward for such exceptional talents; but honors and high position were apparently to be obtained then in only one way, a path full of bitterness and humiliation to the spirit of Hermite, who had all examinations *en horreur*. It was indeed necessary for him to descend from his lofty mathematical speculations which were opening up new fields of research and occupy himself at the age of twenty-four with the petty and irksome details of preparation for the examination leading to the degrees of bachelor of letters and of science, the *licence* and the *agrégation*.

* J. Pierpont, "Early history of Galois's theory of equations." BULLETIN, 2d Series, vol. 4, p. 335.

The first three he passed successfully. He still had the last and most trying of all before him when he fortunately received the position of examinateur d'admission and répétiteur d'analyse at the Ecole Polytechnique in 1848. These five years 1843–1848 were the Sturm und Drang period of Hermite's otherwise placid life. His appointment at the Ecole Polytechnique removed him from his trying and painful position apart from the university world, and placed him in regular standing among his friends and contemporaries. Although his advance was still by no means rapid, others with less talents receiving preference, the doubts and uncertainties which had beset his future and disturbed his meditations now melted away, and Hermite entered on his long and peaceful career with its course clearly marked out.

We close this brief sketch with a few important dates. From 1848 to 1850 he substituted at the Collége de France in place of Libri. In 1856 he was elected to the Institut. In 1862 a place as maître de conférence was created for him at the Ecole Normale. In the following year he was appointed to the responsible position of examinateur de sortie at the Ecole Polytechnique. In 1869 he became professor at this school and in 1870 at the Sorbonne. He was thus forty-seven years of age when he at length reached a position which his extraordinary talents had long claimed as their due. The professorship at the former institution he held till 1876; at the latter he remained in active service till 1897. Hermite was therefore 75 years of age when he closed his academic career.

As a lecturer at the Sorbonne, Hermite achieved a world-wide reputation. Nearly all the present generation of French mathematicians received instruction at his hands and the affectionate veneration of his former pupils was touchingly manifested on the occasion of his jubilee in 1892. Here is the tribute of one of his most gifted pupils, E. Borel: "C'est à la Sorbonne que j'ai suivi les leçons d'Hermite; c'est là que j'ai entendu cette parole si vivante exposer avec respect à la fois et avec amour les belles vérités de l'analyse. C'était un grand prêtre de la divinité du nombre qui nous en dévoilait les mystères redoutables et sacrés. Les questions les plus arides, les calculs en apparence les plus ingrats se transfiguraient, tant il avait l'intuition de leurs secrètes beautés. Quelques-uns peut-être ont eu, autant qu'Hermite, le pouvoir de faire comprendre et admirer les mathématiques ; nul n'a su les faire aimer aussi profondément que lui."*

II.

Let us now turn our attention to the volume under review. This in one respect will certainly prove the most interesting one in Hermite's works. In fact, it is in these early memoirs that we may observe his wonderful genius unfold and mature; it is here that we see the germs of those fruitful notions and principles evolve with which Hermite enriched the theory of numbers, the theory of invariants and of the elliptic and abelian functions. Our astonishment and admiration grow apace as we study the contents of the 37 memoirs which make up this volume and bring us down to the year 1858. At the beginning of his career of research, Hermite seems to have actually suffered from a wealth of ideas; they overwhelm him to such an extent that he cannot take the time to develop them in order and with leisure. Rapid sketches of the main results and principles, hasty aperçus of broad horizons and fruitful fields to be exploited later, are characteristics of many of these first papers.

At the period when Hermite as a young man of twenty began to strike out for himself, the arithmetic theory of the higher forms was just beginning to be studied. Gauss had already taken the first step in the *ternary* quadratic forms when treating the composition of binary quadratic forms in the Disquisitiones Arithmeticæ. Dirichlet and Eisenstein were ably carrying on the work, when Hermite appeared upon the scene with several new and powerful principles. Chief of these are: (1) The introduction of forms with *variable* coefficients; (2) the determination and systematic employment of the upper limits of the minimum values that certain forms can take on when the variables are restricted to integral values. By their aid Hermite attacked with marked success two fundamental problems in the theory of forms, viz: the proof that the number of classes of forms of a given kind is finite, and the determination of all the linear substitutions which leave a given form unchanged. The particular forms that Hermite here studied

* For further details we refer to the following:
G. Darboux, "La vie et l'oeuvre de Charles Hermite," *La Revue du Mois*, Paris, 1906 ; vol. 1, p. 37.
E. Picard, preface to the volume under review.
M. Noether, "Charles Hermite." *Math. Annalen*, vol. 55, p. 337.

are the quadratic forms in n variables, bilinear conjugate imaginary forms, and forms which are the product of n linear factors. While studying these profound questions, many are the applications he makes en route to problems of the most diverse nature. Such are, for example, the following: decomposition of primes into factors formed of the roots of unity; the generalization of continued fractions, leading to a criterion for cubic irrationalities, and also affording a simultaneous approximation to n given numbers; the demonstration that an analytic function of n variables cannot have more than $2n$ periods; the determination of the maximum number of fundamental units in cyclotomic bodies, the finiteness of the number of independent algebraic irrationalities belonging to the same discriminant, and the decomposition of a number as the sum of four squares. Perhaps the most notable of all is his application of the theory of quadratic forms to the theorems of Sturm and Cauchy relative to the number of roots of an algebraic equation in a given region.

It was in the theory of numbers that Hermite first won lasting fame, and the papers on this subject not only form the larger part of the present volume, but they also have a very real importance in the problems of the present. Have we not seen what wonderful developments they were capable of in the hands of Minkowski, and are not the bilinear forms in conjugate imaginaries playing today a prominent role in automorphic functions and linear groups?

However, it was not as a disciple of Gauss and Dirichlet that Hermite made his debut in the mathematical world, but as a zealous student of the writings of Jacobi. Sixteen years before (1827) Jacobi, then an obscure privat docent at Königsberg, penned those lines, now historic, to the venerable Legendre: "Un jeune géomètre ose vous presenter quelques découvertes faites dans la théorie des fonctions elliptiques, auxquelles il a été conduit par l'étude assidue de vos beaux écrits. . . ."

No doubt with these circumstances in mind, Liouville advised Hermite to communicate some of his results to Jacobi, who now was at the zenith of his fame at the University of Königsberg. The reception and generous recognition which this letter received at the hands of Jacobi encouraged Hermite to give a further account of his researches in a second letter. Jacobi thought so highly of them that they were published in his Opuscula Mathematica as well as in *Crelle's Journal* (volume 32, 1846). The first letter was indeed a tour de force for a young man scarce twenty years of age. In 1832 Jacobi had at last succeeded in discovering the inversion of the ultraelliptic integrals and two years later in his paper "De functionibus duarum variabilium," etc., he had given a few meager indications relative to their division and transformation. This was all that was then known. The whole subject was involved in obscurity and but few mathematicians of that day had any knowledge of it whatever. Jacobi might well be surprised therefore on receiving Hermite's letter containing a complete solution of the problem of division. His second letter treats the question of the transformation of these functions, but with less success. The letter is, however, noteworthy as containing the germs of the modern theory of theta functions of one variable, of order n and rational characteristics, and their application to the transformation theory of the elliptic functions.

Jacobi's reply to this letter terminates with the following noble and generous passage: "Ne soyez pas fâché, monsieur, si quelques-unes de vos découvertes se sont rencontrées avec mes anciennes recherches. Comme vous dûtes commencer par où je finis, il y a nécessairement une petite sphère de contact. Dans la suite si vous m'honorez de vos communications, je n'aurai qu'à apprendre."

The problem of transformation of the ultraelliptic functions left unfinished, as just remarked, required for its solution the introduction of new elements, the theta functions in two variables and the arithmetic properties of the abelian group defined by the linear transformation of the four periods. The former was furnished by the papers of Goepel and Rosenhain (1847–1851), the latter by Hermite's own long researches in the theory of numbers. Thus ten years later, in 1855, Hermite was able to give the complete solution of the problem in a memoir which he regarded in later years with just pride. Another important application of the theory of numbers is Hermite's determination of the twenty-fourth root of unity which enters in the linear transformation of the thetas, and whose value until then was unknown. This forms the last paper in the present volume; it is moreover the forerunner of his epoch-making memoirs on the elliptic modular functions, and their application to the solution of the quintic.

To complete our picture of the rich contents of this first volume we have yet to mention Hermite's early contributions to the theory of invariants. This theory which Cayley and Sylvester had just begun to develop attracted him at first by its numerous and important relations with the higher arithmetic. Indeed, in both theories the linear transformations are of prime importance and the first invariants discovered are the determinants of quadratic forms. The present volume contains three great memoirs on binary forms, from the years 1854, 1856; their most interesting and original feature is the application Hermite makes of the principles he had already elaborated in the arithmetic theory of forms. In the profusion of results given in these memoirs we note the law of reciprocity, the existence of quadratic covariants for forms of all degrees except the biquadratic, the introduction of canonical forms, the complete system of forms for the quintic, the discovery of skew invariants, criteria for the reality of the roots of the quintic in terms of its invariants, the sextic resolvent of the quintic and the reduction of elliptic differentials of the first species to the canonical form now known as Weierstrass's.

In these investigations we see Hermite attach a particular importance to the forms of fifth degree. In fact at this time he was carrying on several lines of investigations which were to culminate in his memorable solution of the quintic. Two we have already mentioned; a third we have deferred until the present moment. Like so many of the foremost of his predecessors Hermite had been very early attracted to study the solution of the general equation of fifth degree. While still a pupil at Louis le Grand and ignorant of the work of Abel and Galois, he had published a demonstration of the impossibility of its solution by radicals, which now forms the second paper in the present volume. That Hermite still had this problem in mind we know from a letter of Borchardt, 1847. Thus when Puiseux published his memoir, 1850, on the permutation of the roots of an algebraic equation around the branch points, Hermite immediately made use of these results to develop the notion of the monodromic group of the equation and saw its application to the equations of division and transformation of the elliptic functions, 1851.

A few miscellaneous remarks in closing. As noticed above, the volumes are to appear in royal octavo and not in the customary quarto form. This first volume is furnished with a portrait of Hermite, at about the age of twenty-five; and has a preface by Picard on Hermite's scientific work. The editor gracefully acknowledges the assistance rendered by the late Professor Stouff in checking and correcting by laborious computations the results of some of the memoirs. Many of the papers bear no date and none state the pages of the volume in which they originally appeared. A moment's reflection will convince the editor how important either of these data may be. Hermite sometimes refers in a general way to some of his earlier results and it would greatly help the reader if precise references were given to the pages of the present volume. Let us illustrate. In a paper on the equation for secular inequalities Hermite (page 481), refers to certain forms he introduced "comme on pourra le voir dans un de mes Mémoires publiés dans le *Journal de Crelle*, t. 47." The reader unfamiliar with Hermite's work will naturally turn to the table of contents to find the place of this memoir in the present volume. Here he will make the unpleasant discovery that the titles of the memoirs are unaccompanied by the name and volume of the journals in which they appeared. The reader will therefore have to turn to one title page after another, until he comes to page 193 where he meets a memoir from this volume of *Crelle*. Here he finds nothing about these forms. He looks further and discovers that there is another memoir also in *Crelle* 47. This is a long paper and it takes some time to ascertain that the forms in question are not there. He may now learn that there is still a third memoir in this same volume of *Crelle*, and here on page 237 he may at last find what he so long searched for.

Again, when Hermite refers to the results of others, it would often help the reader if exact reference were given to their collected works. For example, Hermite refers, page 484, to a paper of Jacobi's in *Crelle*, volume 34. This is a misprint, as the paper is to be found in volume 36, page 97, or in Jacobi's Werke, volume 2, page 173. Or again, on page 380, Hermite refers to results "que j'ai indiquées dans le Journal de M. Thomson." As relatively few persons will know what journal is meant, why not add a footnote that it is the *Cambridge and Dublin Mathematical Journal*, and give the precise reference to its place in the present volume, viz., page 301? On pages 79 *seq.* we meet a sign which certainly will be unintelligible to some readers. It is Cauchy's symbol for the residue of an analytic function, and is, we believe, entirely obsolete now.

We have noted the following misprints: Page 165, 8 lines from top, for 46 read 40; page 315, 10 lines from top, for *foudent* read *fondent;* page 332, last line, letter missing; page 380, for *Thompson* read *Thomson*; page 449, a hyphen is missing in next to the last line, also capitalize d; page 473, for *Veierstrass* read *Weierstrass;* page 481, heading of page is wrong; page 498, 8 lines from bottom; for *snr* read *sur*.

JAMES PIERPONT.

PROJECTIVE DIFFERENTIAL GEOMETRY.

Projective Differential Geometry of Curves and Ruled Surfaces. By E. J. WILCZYNSKI. Leipzig, B. G. Teubner, 1906. viii + 295 pages.

THE present volume is the amplification and systematic development of the ideas originally presented in various papers by the author, published in the *American Journal*, *Transactions of the* AMERICAN MATHEMATICAL SOCIETY and *Mathematische Annalen.*

The work begins with a very brief resumé of the ideas of continuous groups, followed by a synopsis of the transformations of linear homogeneous differential equations, wherein Stäckel's theorem regarding the form of the transformations which leave such an equation invariant is generalized to apply to a simultaneous system of such equations. A fairly full discussion is given to the invariants and covariants of a single linear equation. After showing that every transformation which leaves the equation invariant is of the form

$$y = \lambda(x)\eta, \quad x = f(\xi),$$

the first transformation alone is treated at length, the functions of p_i, $p_i^{(k)}$ which remain invariant being designated as seminvariants; those of p_i, $p_i^{(k)}$, $y^{(l)}$ being called semi-covariants. By the second transformation, a function Ω of x is said to be invariant of weight m if

$$\Omega(x) = \frac{1}{(f')^m}\,\Omega(\xi).$$

An early application of these ideas is the derivation of the canonical form of the equation, in which the terms containing $y^{(n-1)}$ and $y^{(n-2)}$ are absent. The Lagrange adjoints are dis-