ON THE SHORTEST DISTANCE BETWEEN CONSECUTIVE STRAIGHT LINES.

BY MR. JOSEPH LIPKE.

CERTAIN well-known geometric results concerning space curves and surfaces have been obtained by a discussion of the shortest distance between consecutive positions of a straight line moving continuously in space. These results have been gained by a discussion (recapitulated in §1) of the *numerator* of the expression for the shortest distance.* It is the purpose of this paper to complete the discussion by examining (§§ 2–3) the *denominator* of the distance expression, placing special emphasis upon the conditions that the distance be an infinitesimal of the *second* order, and upon a geometric interpretation of this case.

§ 1. *Brief Discussion of the Numerator.*

The equations of the straight line are

$$(1) \qquad \begin{aligned} x &= az + p, \\ y &= bz + q, \end{aligned} \quad \text{or} \quad \begin{aligned} x &= a(t)z + p(t), \\ y &= b(t)z + q(t), \end{aligned}$$

where a, b, p, q are analytic functions of a single variable t. We define the consecutive line by the equations

$$(2) \qquad \begin{aligned} x &= a(t + dt)z + p(t + dt), \\ y &= b(t + dt)z + q(t + dt), \end{aligned}$$

or

$$x = \left(a + a'dt + a''\frac{dt^2}{2\,!} + \cdots\right)z + \left(p + p'dt + p''\frac{dt^2}{2\,!} + \cdots\right),$$

$$y = \left(b + b'dt + b''\frac{dt^2}{2\,!} + \cdots\right)z + \left(q + q'dt + q''\frac{dt^2}{2\,!} + \cdots\right),$$

where $a' = da/dt$, $a'' = d^2a/dt^2$, $\cdots$. The formula for the shortest distance between lines (1) and (2) is given by †

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* Koenigs : Géométrie réglée: *Annales de la Faculté des Sciences de Toulouse*, vol. 6, pp. 38–40, 61–63.
Joachimsthal : Anwendungen der Diff. und Int. Rechnung, etc., pp. 182–184.
Knoblauch : Einleitung in die allgemeine Theorie der krummen Flächen, pp. 104–106.

† Laurent : Traité d'analyse, vol. 2, pp. 298–303.

$$(3)\quad d=\frac{\left(p'dt+p''\frac{dt^2}{2!}+\cdots\right)\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)-\left(q'dt+q''\frac{dt^2}{2!}+\cdots\right)\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)}{\sqrt{\left\{\left[b'dt+b''\frac{dt^2}{2!}+\cdots\right]^2+\left[a'dt+a''\frac{dt^2}{2!}+\cdots\right]^2+\left[a\left(b'dt+b''\frac{dt^2}{2!}+\cdots\right)-b\left(a'dt+a''\frac{dt^2}{2!}+\cdots\right)\right]^2\right\}}}$$

and expanding, we get

$$(4)\quad d=\frac{\begin{vmatrix}b'&a'\\q'&p'\end{vmatrix}dt^2+\left\{\frac{1}{2!1!}\begin{vmatrix}b''&a''\\q'&p'\end{vmatrix}+\frac{1}{1!2!}\begin{vmatrix}b'&a'\\q''&p''\end{vmatrix}\right\}dt^3+\left\{\frac{1}{3!1!}\begin{vmatrix}b'''&a'''\\q'&p'\end{vmatrix}+\frac{1}{2!2!}\begin{vmatrix}b''&a''\\q''&p''\end{vmatrix}+\frac{1}{1!3!}\begin{vmatrix}b'&a'\\q'''&p'''\end{vmatrix}\right\}dt^4+\left\{\frac{1}{4!1!}\begin{vmatrix}b^{\text{iv}}&a^{\text{iv}}\\q'&p'\end{vmatrix}+\frac{1}{3!2!}\begin{vmatrix}b'''&a'''\\q''&p''\end{vmatrix}+\frac{1}{2!3!}\begin{vmatrix}b''&a''\\q'''&p'''\end{vmatrix}+\frac{1}{1!4!}\begin{vmatrix}b'&a'\\q^{\text{iv}}&p^{\text{iv}}\end{vmatrix}\right\}dt^5+\cdots}{\sqrt{\left[\left\{b'^2+a'^2+(ab'-ba')^2\right\}dt^2+\left\{\frac{2}{1!2!}b'b''+\frac{2}{1!2!}a'a''+\frac{2}{1!2!}(ab'-ba')(ab''-ba'')\right\}dt^3+\left\{\frac{2}{1!3!}b'b'''+\frac{1}{2!2!}b''^2+\frac{2}{1!3!}a'a'''+\frac{1}{2!2!}a''^2+\frac{2}{1!3!}(ab'-ba')\times(ab'''-ba''')+\frac{1}{2!2!}(ab''-ba'')^2\right\}dt^4+\cdots\right]}}$$

or briefly,

$$(5)\quad d=\frac{N}{D}=\frac{[2]dt^2+[3]dt^3+[4]dt^4+[5]dt^5+\cdots}{\sqrt{\{[\bar{2}]dt^2+[\bar{3}]dt^3+[\bar{4}]dt^4+[\bar{5}]dt^5+\cdots\}}}.$$

It is at once evident that, in general, d is an infinitesimal of the first order; hence

THEOREM I. *The shortest distance between two consecutive generators of a skew surface, is, in general, an infinitesimal of the first order.*

If $[2]\equiv\begin{vmatrix}b'&a'\\q'&p'\end{vmatrix}=0$, $[3]$, which is $\frac{1}{2}d\,[2]/dt$, also vanishes, but $[4]$ does not vanish; then d becomes an infinitesimal of the third order. Now the condition that line (1) moves tangent to the space curve

$$(6)\qquad x=a\psi+p,\quad y=b\psi+q,\quad z=\psi$$

through every point of which one and only one of the lines (1) passes, is that

$$\psi=-\frac{p'}{a'}=-\frac{q'}{b'}\text{ or }\begin{vmatrix}b'&a'\\q'&p'\end{vmatrix}=0;$$

and curve (6) takes the form

$$(7)\qquad x=\frac{pa'-ap'}{a'},\quad y=\frac{qb'-bq'}{b'},\quad z=-\frac{p'}{a'},$$

where $b'p'=a'q'$.

Thus our line generates a developable surface. Hence

THEOREM II. *If the shortest distance between two consecutive positions of a moving line is an infinitesimal of the third order, the line will generate a developable surface, and conversely.*

Again, if $\left|\begin{smallmatrix} b' & a' \\ q' & p' \end{smallmatrix}\right| = 0$ and $\left|\begin{smallmatrix} b'' & a'' \\ q'' & p'' \end{smallmatrix}\right| = 0$ simultaneously, we have, integrating these, the two sets of solutions (*i*) $b = ca + c_1$, $q = cp + c_2$ and (*ii*) $p = ka + k_1$, $q = kb + k_2$ for either of which every $\left|\begin{smallmatrix} b^{(n)} & a^{(n)} \\ q^{(m)} & p^{(m)} \end{smallmatrix}\right|$ vanishes identically, hence N vanishes identically, and d becomes zero. Now, under conditions (*i*) line (1) becomes

$$(8) \qquad x = az + p, \quad y = (ca + c_1)z + cp + c_2$$

and curve (7) takes the form

$$(9) \quad x = \frac{pa' - ap'}{a'}, \quad y = \frac{c(pa' - ap') + c_2 a' - c_1 p'}{a'}, \quad z = -\frac{p'}{a'},$$

a curve lying in the plane $y = cx + c_1 z + c_2$, hence the line moves tangent to a plane curve. Under conditions (*ii*) line (1) becomes

$$(10) \qquad x = az + ka + k_1, \quad y = bz + kb + k_2$$

and curve (7) takes the form

$$(11) \qquad x = k_1, \quad y = k_2, \quad z = -k,$$

a point, and the line generates a cone. Hence

THEOREM III. *If the shortest distance between two consecutive positions of a moving line is identically zero, the line will move tangent to a plane curve or generate a cone, and conversely.*

We also note that $N = 0$ when $a' = 0, b' = 0$, or $a =$ const., $b =$ const., *i. e.*, our line moves parallel to itself and generates a cylinder; but this needs further discussion since D also vanishes here.

Finally, in any case, N is always an infinitesimal of even order.*

THEOREM IV. *For special lines of a surface (skew or developable), the shortest distance between two consecutive generators may be an infinitesimal of higher order than the third, but always of odd order.*

* Zindler: Liniengeometrie mit Anwendungen: Zweiter Teil, p. 13.

§ 2. *Discussion of the Denominator.*

The denominator of the expression for d is

$$(12)\quad \begin{aligned} D &= \sqrt{\{[\bar{2}]dt^2 + [\bar{3}]dt^3 + [\bar{4}]dt^4 + \cdots\}} \\ &= \sqrt{\{[b'^2 + a'^2 + (ab' - ba')^2]dt^2 + [b'b''} \\ &\quad \overline{+ a'a'' + (ab' - ba')(ab'' - ba'')]dt^3 + [\tfrac{1}{3}b'b'''} \\ &\quad \overline{+ \tfrac{1}{4}b''^2 + \tfrac{1}{3}a'a''' + \tfrac{1}{4}a''^2 + \tfrac{1}{3}(ab' - ba')(ab''' - ba''')} \\ &\quad \overline{+ \tfrac{1}{4}(ab'' - ba'')^2]dt^4 + \cdots\}}. \end{aligned}$$

We have

$$[\bar{3}] = \tfrac{1}{4}\frac{d}{dt}[\bar{2}].$$

Hence

1) $[\bar{3}]$ will vanish identically whenever $[\bar{2}]$ vanishes identically, and

2) In general, D is an infinitesimal of the first order.

Let us find the solution of the differential equation

$$[\bar{2}] \equiv b'^2 + a'^2 + (ab' - ba,)^2 = 0.$$

The only real solutions of this equation are at once seen to be $b' = 0$, $a' = 0$, or $b =$ const., $a =$ const.; these cause D to vanish identically. Hence

3) $b =$ const., $a =$ const. (the real solutions of $[\bar{2}] = 0$) cause D to vanish identically.

Now $[\bar{2}] = 0$ has imaginary solutions. To find these let us write the equation in the equivalent forms

$$\text{(i)}\qquad (a^2+1)\left(\frac{db}{dt}\right)^2 + (b^2+1)\left(\frac{da}{dt}\right)^2 - 2ab\frac{db}{dt}\frac{da}{dt} = 0,$$

or

$$\text{(ii)}\qquad b^2 - 2ab\frac{db}{da} + (a^2+1)\left(\frac{db}{da}\right)^2 + 1 = 0,$$

or

$$\text{(iii)}\qquad b = a\frac{db}{da} \pm i\sqrt{\left(\frac{db}{da}\right)^2 + 1}.$$

This differential equation is in Clairaut's form, and its integral is

$$(13)\qquad b = ca \pm i\sqrt{c^2+1}. \quad (c = \text{constant.})$$

This gives rise to the system of equations

$$b' = ca', \quad b'' = ca'', \quad b''' = ca''', \cdots, \quad b^{(n)} = ca^{(n)}, \cdots$$

and it is easily seen from (3) that D vanishes identically. Hence

4) The general solution $b = ca \pm i\sqrt{c^2+1}$ of the differential equation $[\bar{2}] = 0$ makes D vanish identically.

Now the differential equation $[\bar{2}] = 0$ is one of second degree and therefore has a singular solution. The latter is most readily found by solving the form (ii) as a quadratic equation in db/da, and setting the discriminant equal to zero. The singular solution is found to be

$$(14) \qquad a^2 + b^2 + 1 = 0.$$

This solution (14) causes $[\bar{2}]$ and $[\bar{3}]$ to vanish and, as can easily be found with a little calculation, makes $[\bar{4}] = -b'^4/4a^4$, *i. e.*, does not cause $[\bar{4}]$ to vanish. Hence

5) The singular solution $a^2 + b^2 + 1 = 0$ of the differential equation $[\bar{2}] = 0$ reduces D to an infinitesimal of the second order.

Again if $[\bar{4}]$ vanishes in addition to the vanishing of $[\bar{2}]$ through its singular solution, we must have, since $[\bar{4}] = -b'^4/4a^4 = 0$, $b' = 0$, or $b =$ const., which taken in conjunction with $a^2 + b^2 + 1 = 0$, gives also $a =$ const.; but $a =$ const., $b =$ const., cause D to vanish identically. Hence

6) The singular solution $a^2 + b^2 + 1 = 0$ of $[\bar{2}] = 0$ taken simultaneously with $[\bar{4}] = 0$ causes D to vanish identically.

Thus, we finally have

7) D is in general an infinitesimal of the first order; if $a^2 + b^2 + 1 = 0$, D is an infinitesimal of the second order; if $b = ca \pm i\sqrt{c^2+1}$, or if $b =$ const., a const., D is identically zero.

§ 3. *Geometric Interpretations.*

We have

$$(15) \qquad d = \frac{N}{D} = \frac{[2]\,dt^2 + [3]\,dt^3 + [4]\,dt^4 + \cdots}{\sqrt{\{[\bar{2}]\,dt^2 + [\bar{3}]\,dt^3 + [\bar{4}]\,dt^4 + \cdots\}}}.$$

If $[\bar{2}] = 0$, through its general solution $b = ca \pm i\sqrt{c^2+1}$

and also $[2] \equiv \left|{}^{b'a'}_{q'p'}\right| = 0$, then $q'/p' = b'/a' = c$, or $q' = cp'$, and $q = cp + c_2$; but then both D and N vanish identically, and d takes the indeterminate form $0/0$. But this indeterminacy is easily resolved, for the conditions $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, are only a special case of $b = ca + c_1$, $q = cp + c_2$, the conditions that the line move tangent to a plane curve. Thus d is zero. The curve (6) becomes

$$(16)\quad x = \frac{pa' - ap'}{a'},\ y = \frac{a(pa'-ap') + c_2a' \mp i\sqrt{c^2+1}\,p'}{a'},\quad z = -\frac{p'}{a'},$$

an imaginary curve lying in the imaginary plane

$$y = cx \pm i\sqrt{c^2+1}\,z + c_2.$$

THEOREM VI. *If a continuously moving straight line* $x = az + p$, $y = bz + q$ *obeys the conditions* $b = ca \pm i\sqrt{c^2+1}$, $q = cp + c_2$, *it will move tangent to an imaginary curve in an imaginary plane, the distance between two consecutive tangent lines being zero.*

Again, if $[\bar{2}]$ vanishes through its singular solution $a^2 + b^2 + 1 = 0$, then by 7) D is an infinitesimal of the second order, and if $[2]$ does not vanish, N is also an infinitesimal of the second order, and thus d is finite. Now the relation $a^2 + b^2 + 1 = 0$ expresses that the sum of the squares of three quantities proportional to the direction cosines of our line, is equal to zero, which property is the distinguishing characteristic of a minimal straight line of space. Hence

THEOREM VII. *The shortest distance between two consecutive generators of a skew surface generated by a continuously moving minimal straight line, is finite.*

If in addition to the vanishing of $[\bar{2}]$ through its singular solution $a^2 + b^2 + 1 = 0$, $[2] = \left|{}^{b'a'}_{q'p'}\right|$ also vanishes, N is an infinitesimal of the fourth order, for we have

$$a^2 + b^2 + 1 = 0 \quad \text{and} \quad \begin{vmatrix} b' & a' \\ q' & p' \end{vmatrix} = 0$$

or

$$aa' + bb' = 0 \quad \text{and} \quad \frac{b'}{a'} = \frac{q'}{p'} = -\frac{a}{b}.$$

$$\therefore q' = -\frac{a}{b}p' \quad \text{or} \quad q = -\int \frac{a}{b}p'dt\,.$$

$$\therefore b = \pm i\sqrt{a^2+1} \quad \text{and} \quad q = \pm i\int \frac{a}{\sqrt{a^2+1}} p'dt.$$

With these conditions we can easily calculate

$$\begin{vmatrix} b'' & a'' \\ q'' & p'' \end{vmatrix} = \frac{a'}{b^3}(a''p' - a'p'') \neq 0,$$

hence [4] does not vanish, *i. e.*, N is an infinitesimal of the fourth order. Thus d is an infinitesimal of the second order. The line moves tangent to a minimal curve in space. The equations of the line are

$$(17) \qquad x = az + p, \quad y = \pm i\sqrt{a^2+1}\, z \pm i\int \frac{a}{\sqrt{a^2+1}} p'dt,$$

and those of the minimal curve to which the line moves tangent are, from (6),

$$(18) \qquad x = \frac{pa' - ap'}{a'}, \qquad z = -\frac{p'}{a'},$$

$$y = \mp i\frac{\sqrt{a^2+1}\,p'}{a'} \pm i\int \frac{a}{\sqrt{a^2+1}} p'dt.$$

Hence

THEOREM VIII. *The shortest distance between any two consecutive generators of the tangential surface to a minimal curve in space, i. e., of a minimal developable, is an infinitesimal of the second order; and conversely, if a line moves so that the shortest distance between any two consecutive positions is an infinitesimal of the second order, it will generate a minimal developable.*

The converse as stated in Theorem VIII is easily deduced from the above discussion, for d is an infinitesimal of second order only if $[2] = 0$, $[4] \neq 0$, $[\overline{2}] = 0$ (through its singular solution) and $[\overline{4}] \neq 0$.

From equations (18) we have

$$x' = \frac{a(p'a'' - a'p'')}{a'^2}, \quad y' = \frac{\pm i\sqrt{a^2+1}(p'a'' - a'p'')}{a'^2},$$

$$z' = \frac{p'a'' - a'p''}{a'^2}$$

and hence, the equations of the minimal curve may be written

$$(19)\quad x=\int\frac{a(p'a''-a'p'')}{a'^2}dt,\quad y=\pm i\int\frac{\sqrt{a^2+1}(p'a''-a'p'')}{a'^2}dt,$$

$$z=\int\frac{p'a''-a'p''}{a'^2}dt.$$

Now set $a=(1-\tau^2)/2\tau$, where τ is an arbitrary function of t,

$$\therefore \pm i\sqrt{a^2+1}=\frac{i(1+\tau^2)}{2\tau},$$

and set $(p'a''-a'p'')/a'^2dt=\tau F(\tau)d\tau$, where $F(\tau)$ is an arbitrary function of τ. Then equations (19) become

$$(20)\quad x=\frac{1}{2}\int(1-\tau^2)F(\tau)d\tau,\quad y=\frac{i}{2}\int(1+\tau^2)F(\tau)d\tau,$$

$$z=\int\tau F(\tau)d\tau,$$

the well-known equations of the minimal curve.

Using equations (20), we have

$$[2]=0,\ [3]=0,\ [\bar{2}]=0,\ [\bar{3}]=0,\ [4]=\frac{-iF(\tau)}{12\tau^2},\ [\bar{4}]=-\frac{1}{4\tau^4}$$

and

$$(21)\quad d=\frac{\dfrac{-iF(\tau)}{12\tau^2}d\tau^4+\cdots}{\sqrt{\left\{-\dfrac{1}{4\tau^4}d\tau^4+\cdots\right\}}}=-\frac{1}{6}F(\tau)d\tau^2+\cdots$$

Using equations (19), we have $[4]=-a'/12b^3\times(p'a''-a'p'')$, $[\bar{4}]=-b'^4/4a^4$ and

$$(22)\quad d=\frac{i}{6}\frac{p'a''-a'p''}{ba'}dt^2+\cdots=\frac{p'a''-a'p''}{6a'\sqrt{a^2+1}}dt^2+\cdots.$$

Finally we may have $[\bar{2}]=0$ through its singular solution $a^2+b^2+1=0$ and $[\bar{4}]=0$, *i. e.*, $a=$ const., $b=$ const. Here both N and D are zero, and hence d has the indeterminate form $0/0$. Since $\psi=-p'/a'=-p'/0=\infty$, the curve (6) becomes $x=\infty$, $y=\infty$, $z=\infty$, *i. e.*, our lines all pass through the same point at infinity; but for the purpose in hand we cannot consider the cylinder as a special case of the cone, *i. e.*, a cone whose vertex is at infinity, for when $a =$ const., $b =$ const., the conditions that the line generates a cone, viz., $p = ka + k_1$, $q = kb + k_2$ become $p =$ const., $q =$ const. and the two consecutive generators actually coincide. Thus, we cannot say, that the shortest distance between two consecutive generators of a cylinder is zero, *i. e.*, that the two generators actually intersect. There is no shortest distance between two such lines; they are everywhere equally distant. Hence, to find the distance between two consecutive parallel lines, we shall have to use the formula for the distance of a point from a line. It is easily seen that, in general, this distance is an infinitesimal of the first order; it is zero only if the two consecutive lines coincide; it is infinite when $a^2 + b^2 + 1 = 0$. Hence

THEOREM IX. *The distance between two consecutive generators of a cylinder is, in general, an infinitesimal of the first order; if the generator is a minimal straight line, the distance is infinite.*

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NOTE ON THE COMMUTATOR OF TWO OPERATORS.

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(Read before the American Mathematical Society, April 27, 1907.)

THERE is a confusing lack of uniformity with respect to the use of the term commutator. The present note aims to exhibit this fact and to point out some of its sources in the hope that these data may tend towards greater uniformity in the use of this term and also make its various meanings less confusing to the reader.

The operation now known as the commutator of two operators was used for a long time in the development of group theory before it received a special name. It is frequently employed, in various forms, in Jordan's Traité des substitutions, and its elegant application in the study of direct products was recognized by Hölder* and others. The first paper which deals with the

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*Hölder, *Math. Annalen*, vol. 34 (1889), p. 35. It should be noted that the reference 91) in Encyklopädie der mathematischen Wissenschaften, vol. 1, p. 219, should be to this article instead of to the later one in vol. 43.