BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

APPLICATION OF A DEFINITE INTEGRAL INVOLVING BESSEL'S FUNCTIONS TO THE SELF-INDUCTANCE OF SOLENOIDS.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society, December 29, 1905).

THE construction in the Physical Laboratory of Clark University of two primary standards of self-inductance by Dr. J. G. Coffin and his development of satisfactory formulas for the calculation of their values* led the writer to attempt to find an expression of a different sort for this quantity, namely the self-inductance of a current sheet in the form of a circular cylinder, practically represented by a wire wound in a close solenoid of a single layer.† All the methods that have been previously used have involved elliptic integrals or developments in spherical harmonics. A quite different mode of approaching the problem was suggested by a paper by Nagaoka,‡ in which use is made of the following integral given by H. Weber § for the value of the potential of a circular disk of radius a and of unit surface density at a point at a distance z from its plane and r from the axis of symmetry :

$$(1) \qquad V = 2\pi a \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_0(\lambda r) J_1(\lambda a) d\lambda \qquad (z > 0).$$

* J. G. Coffin, *Bulletin* of the Bureau of Standards, vol. 2, No. 1 (June, 1906).

† Since this paper was read several papers by E. B. Rosa and L. Cohen on the *mutual* inductance of such solenoids have appeared in the *Bulletin* of the Bureau of Standards.

‡ Nagaoka, "On the potential and lines of force of a circular current," *Phil. Mag.*, vol. 6, p. 19, 1903.

§ H. Weber, "Ueber die Bessel'schen Funktionen und ihre Anwendungen auf die Theorie der elektrischen Ströme." *Crelle's Journal*, vol. 75, p. 75.

The magnetic field due to a current I in a close solenoid of m turns per unit of length parallel to the axis, which we will take as the z-axis, is $4\pi mI$, if the solenoid is of infinite length. If not, we can find the effect of the ends by placing at the positive and negative ends disks with surface density $\pm mI$ respectively. We have then for the magnetic potential at points inside the solenoid

$$(2) \qquad \Omega = mI(V_1 - V_2 - 4\pi z),$$

and for the field

$$(3) \qquad H = -\frac{\partial \Omega}{\partial z} = mI\left(4\pi - \frac{\partial V_1}{\partial z} + \frac{\partial V_2}{\partial z}\right),$$

in which V_2 denotes the value obtained by putting in (1) for z the distance z_2 of the point from the negative end of the cylinder, and V_1 the value obtained by putting the distance from the positive end $z_1 = l - z_2$, where l is the length of the solenoid.

Differentiating under the integral sign, since

$$\frac{\partial}{\partial z_2} = -\frac{\partial}{\partial z_1} = \frac{\partial}{\partial z},$$

we find the flux through a cross-section of the cylinder by multiplying by $2\pi r dr$ and integrating from 0 to a,

$$(4) \qquad \begin{aligned} 2\pi \int_0^a H r dr = mI \Big\{ & 4\pi^2 a^2 - 4\pi^2 a \int_0^a r dr \int_0^\infty e^{-\lambda z_1} J_0(\lambda r) J_1(\lambda a) d\lambda \\ & - 4\pi^2 a \int_0^a r dr \int_0^\infty e^{-\lambda z_2} J_0(\lambda r) J_1(\lambda a) d\lambda \Big\}. \end{aligned}$$

Now we have

$$\int_0^a r J_0(\lambda r) dr = -\frac{1}{\lambda^2}\int_0^{\lambda a} \xi J_0(\xi) d\xi = \frac{1}{\lambda^2}\left[\xi J_1(\xi)\right]_0^{\lambda a} = \frac{a}{\lambda} J_1(\lambda a),$$

so that, changing the order of integration, we obtain for the flux through the circular section

$$(5) \qquad 4\pi^2 a^2 mI \left\{ 1 - \int_0^\infty \frac{e^{-\lambda z_1}}{\lambda} J_1^2(\lambda a) d\lambda - \int_0^\infty \frac{e^{-\lambda z_2}}{\lambda} J_1^2(\lambda a) d\lambda \right\}.$$

Since the length dz carries the current $mIdz$, multiplying this by the flux just found and integrating for the whole length l we obtain for the self-inductance L

$$(6)\qquad LI^2 = 4\pi^2 a^2 m^2 I^2 \left\{ l - 2\int_0^l dz \int_0^\infty \frac{e^{-\lambda z}}{\lambda} J_1^2(\lambda a) d\lambda \right\}.$$

We may now interchange the order of integration, performing the integration according to z,

$$(7)\qquad L = 4\pi^2 a^2 m^2 \left\{ l + 2\int_0^\infty (e^{-\lambda l} - 1) \frac{J_1^2(\lambda a)}{\lambda^2} d\lambda \right\}.$$

We may arrive more quickly at the result (7) by starting from the fundamental definition of the mutual inductance $M_{xx'}$ of two circular lines situated at distances x and x' from one end of the cylinder, both of which are made to occupy all positions possible, when

$$(8)\qquad L = \int_0^l \int_0^l M_{xx'} dx dx',$$

$$(9)\qquad \begin{aligned} M_{xx'} &= \int\int \frac{ds ds' \cos \epsilon}{r} \\ &= 2\pi a a' \int_0^{2\pi} \frac{\cos \theta d\theta}{\sqrt{(x - x')^2 + a^2 + a'^2 - 2aa' \cos \theta}}. \end{aligned}$$

In order to transform the elliptic integral (9) we make use of the integral of Lipschitz (Gray and Mathews, page 72)

$$(10)\qquad \int_0^\infty e^{-\lambda z} J_0(\lambda \omega) d\lambda = \frac{1}{\sqrt{z^2 + \omega^2}},$$

and the addition formula of Neumann (Gray and Mathews, page 27, 69′)

$$(11)\quad J_0(\lambda\omega) = J_0(\lambda a) J_0(\lambda a') + 2\sum_{n=1}^{n=\infty} J_n(\lambda a) J_n(\lambda a') \cos n\theta,$$

where

$$\omega^2 = a^2 + a'^2 - 2aa' \cos \theta,$$

which being multiplied by $\cos \theta$ and integrated from 0 to 2π gives

$$(12)\qquad \int_0^{2\pi} J_0(\lambda\omega) d\theta = 2\pi J_1(\lambda a) J_1(\lambda a').$$

Treating (10) in the same manner, and using this result, we convert (9) into

$$(13) \qquad M_{xx'} = 4\pi^2 aa' \int_0^\infty e^{-\lambda|x-x'|} J_1(\lambda a) J_1(\lambda a') d\lambda.$$

We may now perform the integration with regard to x' and then x, taking care to notice that the cofactor of $-\lambda$ in the exponential must be positive, so that we put $x - x' = z$ when $x > x'$ and $x - x' = -z$ when $x < x'$, giving

$$\int_0^l M_{xx'} dx' = 4\pi^2 aa' \int_0^\infty J_1(\lambda a) J_1(\lambda a') d\lambda \left[\int_0^x e^{-\lambda z} dz + \int_0^{l-x} e^{-\lambda z} dz \right]$$

$$[\cdots] = \frac{1}{\lambda}(2 - e^{-\lambda x} - e^{-\lambda(l-x)}), \quad \int_0^l [\cdots]\, dx = \frac{2}{\lambda}\left(l + \frac{e^{-l\lambda} - 1}{\lambda}\right).$$

$$(14) \qquad \int_0^l \int_0^l M_{xx'} dx dx' = 8\pi^2 aa' \int_0^\infty \left(\frac{l}{\lambda} + \frac{e^{-\lambda l} - 1}{\lambda^2} \right) J_1(\lambda a) J_1(\lambda a') d\lambda.$$

This formula serves for the mutual inductance of two solenoids of the same length but of different diameters, but for our purpose $a = a'$, so that we again obtain (7), since

$$\int_0^\infty \frac{J_1^2(t)}{t} dt = \frac{1}{2}.$$

Let us now introduce the variable $t = \lambda a$ and the constant $z = l/a$ for the ratio of the length to the radius of the solenoid, so that

$$(15) \quad L = 8\pi^2 a^3 m^2 \left[\frac{z}{2} - \int_0^\infty \frac{J_1^2(t)}{t^2} dt + \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt \right].$$

Of these two definite integrals, the first is a constant, whose value we find by the formula given by Nielsen, Handbuch der Theorie der Cylinderfunktionen, page 194, (15) to be $4/3\pi$, while the second is a function of z which we will develop into a power series.

$$(16) \qquad Z = \int_0^\infty e^{-zt} \frac{J_1^2(t)}{t^2} dt.$$

Now we have (Nielsen, page 20, (4)),

$$(17)\quad J_1^2(t)=\sum_0^\infty \frac{(-1)^s(2s+2)!}{s!\,(s+1)!\,(s+1)!\,(s+2)!}\left(\frac{t}{2}\right)^{2s+2},$$

so that

$$(18)\quad Z=\sum_0^\infty \frac{(-1)^s(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\int_0^\infty e^{-zt}t^{2s}dt.$$

The definite integral is now a gamma function, whose value is easily seen to be

$$(2s)!/z^{2s+1}.$$

Thus we obtain

$$(19)\quad Z=\sum_0^\infty \frac{(-1)^s(2s)!\,(2s+2)!}{2^{2s+2}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}},$$

a series which is convergent when $z>2$. We have accordingly the final result

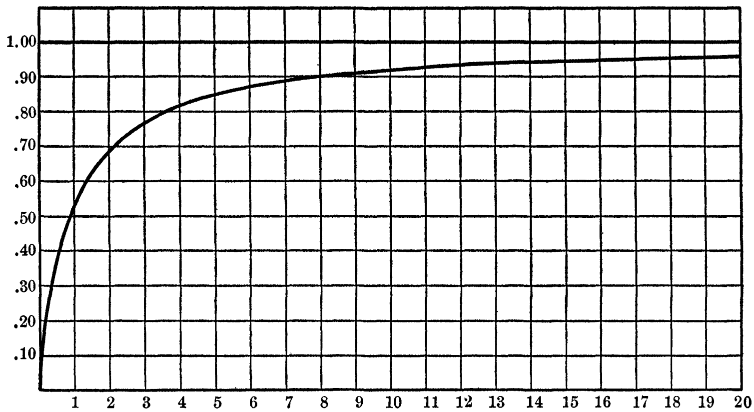
$$(20)\quad \begin{aligned} L &= 4\pi^2a^3m^2\left\{z-\frac{8}{3\pi}+\sum_0^\infty \frac{(-1)^s(2s)!\,(s+2)!}{2^{2s+1}s!\,(s+1)!\,(s+1)!\,(s+2)!}\,\frac{1}{z^{2s+1}}\right\} \\ &= 4\pi^2a^3m^2\left\{z-\frac{8}{3\pi}+\frac{1}{z}-\frac{1}{2z^3}+\frac{5}{8z^5}-\frac{35}{32z^7}+\frac{147}{64z^9}-\frac{693}{128z^{11}}+\frac{14,157}{1,024z^{13}}-\cdots\right\}, \end{aligned}$$

giving the inductance directly as a power series in terms of the ratio of length to radius, when the length of the cylinder is greater than its diameter. The series does not converge rapidly until the length is several times the diameter, but Coffin's formula

$$(21)\quad \begin{aligned} L = 4\pi an^2\Big\{&(N-\tfrac{1}{2})+\frac{z^2}{32}(N+\tfrac{1}{4})-\frac{z^4}{1,024}(N-\tfrac{2}{3}) \\ &+\frac{10z^6}{131,072}(N-\tfrac{109}{120})-\frac{35z^8}{4,194,304}(N-\tfrac{431}{420})\cdots\Big\}, \end{aligned}$$

$$\left(N=\log_e\frac{8a}{l},\ n=lm\right),$$

converges better the shorter the cylinder, so that by one of these formulas the calculation may always be made. Coffin has also given a formula by elliptic integrals, which is convenient when tables are at hand. Mr. Gordon Fulcher has



calculated by the three methods values for some twenty ratios, from which he has constructed the annexed graph of $L/4\pi^2a^3m^2z^2$, that is the factor of correction for the ends of the solenoid. When the length is ten times the diameter, five terms of (20) give nine figures of the result.

CLARK UNIVERSITY,
July, 1907.

ON THE APSIDAL ANGLE IN CENTRAL ORBITS.

BY DR. F. L. GRIFFIN.

(Read in part before the American Mathematical Society, April 27, 1907.)

THERE are two well known laws of central force all of whose trajectories have the same apsidal angle, whatever be the apsidal values of the radius vector, viz., that of Newton and the law that the force varies directly as the distance. For both of these laws the orbits are all conic sections, the apsidal angle being π in the former case, and $\frac{1}{2}\pi$ in the latter. Generally, however, the apsidal angle depends upon the apsidal values of the radius vector.

In this paper are considered only those laws of central force for which the force is a function of the distance, having a finite