THE MAXIMUM VALUE OF A DETERMINANT.

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Hadamard* has shown that the maximum value reached by the modulus of a determinant of order \( n \) the moduli of whose elements do not exceed unity is \( n^{n/2} \). The result is fundamental in Fredholm’s theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway’s work was published. My own was published‡ only in the abstract here reproduced.

"The elements of a determinant being restricted to a variation between the limits \(- a\) and \(+ a\), it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements \(- a\) in the principal diagonal and making all the other elements of the determinant \(+ a\). If we denote such a determinant of order \( n \) when \( a = 1 \) by \( D_n \) and the minor of an element in the \( i \)th row and \( j \)th column by \( A_{ij} \), we have always

\[
A = -(1 + \frac{n-1}{n-3}) \cdot D_{n-1} = + (1 + \frac{n-1}{n-3}) (1 + \frac{n-2}{n-4}) D_{n-2}
\]

\[= \pm \left(1 + \frac{n-1}{n-3}\right) \left(1 + \frac{n-2}{n-4}\right) \cdots \left(1 + \frac{4}{3}\right) \left(1 + \frac{3}{2}\right) \left(1 + \frac{3}{1}\right) D_n\]

\[= \pm (n - 2)2^{n-1}, \text{ since } D_2 = 4.\]

"The sign will be \( \pm \) according as \( n \) is odd or even. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

"For the maximum cubic determinant \( D_n^{(3)} \), we have merely to make all the strata identical and equal to \( D_n^{(2)} \). Its value is \( \pm n! D_n^{(3)} \).

"The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the develop-

opment being cubic determinants of which the plane determinants are the strata. In this quasi-determinant the same rule of signs applies as in the ordinary determinant and we get for the maximum value accordingly

$$\pm D_n^{(2)} n! D_n^{(2)} \alpha^n = \pm n! (D_n^{2})^2.$$ 

Similarly

$$D_n^{(2p)} = \pm (n!)^{p-1} (D_n^{(2)})^p, \quad D_n^{(2p+1)} = \pm (n! D_n^{(2)})^p.$$ 

"In any of these determinants we have at once a formula for the maximum number of positive terms in the development. This is for $D_n^{(q)}$

$$\frac{(n!)^{q-1} \pm D_n^{(q)}}{2}.$$ 

"We use the $\pm$ sign according as $D_n^{(q)}$ is $\pm$." 

A simple example serves to show that the limit reached when the elements are real may be exceeded when the elements are complex. Thus

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4,$$

but

$$\begin{vmatrix} i & 1 & 1 \\ 1 - \frac{1 + i}{\sqrt{2}} & 1 \\ 1 & \frac{1 + i}{\sqrt{2}} & -1 \end{vmatrix} = 2(\sqrt{2} + 1).$$

This explains why Hadamard's limit exceeds mine.

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