THE MAXIMUM VALUE OF A DETERMINANT.

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HADAMARD * has shown that the maximum value reached by the modulus of a determinant of order n the moduli of whose elements do not exceed unity is $n^{n/2}$. The result is fundamental in Fredholm's theory of integral equations.†

The problem, when the elements are required to be real, was studied by Professor Hathaway and myself in 1882. I do not think Mr. Hathaway's work was published. My own was published ‡ only in the abstract here reproduced.

"The elements of a determinant being restricted to a variation between the limits $-a$ and $+a$, it is found that for all determinants whose order is greater than 2, a numerical maximum will be obtained by putting the elements $-a$ in the principal diagonal and making all the other elements of the determinant $+a$. If we denote such a determinant of order n when $a=1$ by D_n and the minor of an element in the ith row and kth column by A_{ik}, we have always $A_{ii}=D_{n-1}$, $A_{ik}=-D_{n-1}/(n-3)$, so that

$$D_n=-\left(1+\frac{n-1}{n-3}\right)\cdot D_{n-1}=+\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)D_{n-2}$$

$$=\pm\left(1+\frac{n-1}{n-3}\right)\left(1+\frac{n-2}{n-4}\right)\cdots\left(1+\frac{5}{3}\right)\left(1+\frac{4}{2}\right)\left(1+\frac{3}{1}\right)D_3$$

$$=\pm(n-2)2^{n-1},\text{ since } D_3=4.$$

"The sign will be $\pm$ according as n is $\substack{\text{odd}\\\text{even}}$. The effect of a change in any element is to lessen the greatest factor leaving the rest the same.

"For the maximum cubic determinant $D_n^{(3)}a^n$, we have merely to make all the strata identical and equal to $D_n^{(2)}$. Its value is $\pm n!\,D_n^{(2)}a^n$.

"The four-dimensional determinant may be looked upon as a sort of determinant of plane determinants, the terms of the devel-

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* *Bulletin des sciences math.*, 1893, p. 240. Pascal, I Determinanti, p. 240.

† *Acta Math.*, vol. 27 (1903), p. 365.

‡ Johns Hopkins University Circular, vol. 2, No. 20 (December, 1882), p. 22.

opment being cubic determinants of which the plane determinants are the strata. In this quasi-determinant the same rule of signs applies as in the ordinary determinant and we get for the maximum value accordingly

$$\pm D_n^{(2)} n\,!\, D_n^{(2)} a^n = \pm\, n\,!\,(D_n^2)^2.$$

Similarly

$$D_n^{(2p)} = \pm\,(n\,!)^{p-1}(D_n^{(2)})^p, \qquad D_n^{(2p+1)} = \pm\,(n\,!\,D_n^{(2)})^p.$$

"In any of these determinants we have at once a formula for the maximum number of positive terms in the development. This is for $D_n^{(q)}$

$$\frac{(n\,!)^{q-1} \pm D_n^{(q)}}{2}.$$

"We use the $\pm$ sign according as $D_n^{(q)}$ is $\pm$."

A simple example serves to show that the limit reached when the elements are real may be exceeded when the elements are complex. Thus

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4,$$

but

$$\begin{vmatrix} i & 1 & 1 \\ 1 & \dfrac{-1+i}{\sqrt{2}} & 1 \\ 1 & \dfrac{1+i}{\sqrt{2}} & -1 \end{vmatrix} = 2(\sqrt{2}+1).$$

This explains why Hadamard's limit exceeds mine.

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