THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and thirty-seventh meeting of the Society was held in New York City on Saturday, February 29, 1908, extending through the usual morning and afternoon sessions. The attendance included the following thirty-four members:

Professor G. A. Bliss, Professor Joseph Bowden, Professor E. W. Brown, Professor F. N. Cole, Professor L. P. Eisenhart, Mr. G. W. Hartwell, Dr. A. M. Hiltebeitel, Professor W. H. Jackson, Professor Edward Kasner, Mr. E. H. Koch, Mr. W. C. Krathwohl, Dr. G. H. Ling, Mr. L. L. Locke, Professor Max Mason, Mr. A. R. Maxson, Professor H. B. Mitchell, Dr. R. L. Moore, Mr. H. W. Reddick, Professor L. W. Reid, Mr. L. P. Siceloff, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Dr. W. M. Strong, Dr. Elijah Swift, Professor H. D. Thompson, Miss M. E. Trueblood, Professor J. M. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor A. G. Webster, Professor H. S. White, Professor J. W. Young.

The President of the Society, Professor H. S. White, occupied the chair, being relieved at the afternoon session by Professor P. F. Smith. The Council announced the election of the following persons to membership in the Society: Mr. E. G. Bill, Yale University; Mr. C. H. Currier, Brown University; Mr. W. S. Pemberton, State Normal School, Edmond, Okla.; Professor S. W. Reaves, University of Oklahoma; Mr. L. L. Silverman, University of Missouri; Mr. W. M. Smith, Lafayette College; Mr. H. W. Stager, University of California. Twelve applications for membership in the Society were received.

The By-Laws of the Society were amended to provide that the appointment of members of the Editorial Committee of the *Transactions* should be made at the April meeting of the Council and that each new member should take office on the following October 1.

The usual informal dinner was arranged for the evening, and was attended by twelve members.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the numerical factors of certain arithmetic forms."

(2) Professor R. D. CARMICHAEL: "On the remainder term in a certain development of $f(a+x)$."

(3) Dr. F. R. SHARPE: "The Lorentzian transformation and the radiation from a moving electron."

(4) Professor VIRGIL SNYDER: Normal curves of genus 6 and their groups of birational transformations."

(5) Professor J. W. YOUNG: "The geometry of chains on a complex line."

(6) Professor J. W. YOUNG: "A fundamental invariant of discontinuous ζ-groups defined by a normal curve of order n in space of n dimensions."

(7) Professor E. B. VAN VLECK: "On non-measurable point sets."

(8) Professor L. E. DICKSON: "On higher congruences and modular invariants."

(9) Professor MAX MASON: "Note on Jacobi's equation in the calculus of variations."

(10) Dr. G. W. HILL: "Subjective geometry."

(11) Professor G. A. BLISS: "A method of deriving Euler's equation by means of an invariant integral."

(12) Professor C. N. HASKINS: "On the second law of the mean."

(13) Professor EDWARD KASNER: "The contact transformations of mechanics."

(14) Professor EDWARD KASNER: "The plane sections of an arbitrary surface."

(15) Professor G. A. MILLER: Note on the periodic decimal fractions."

(16) Dr. F. R. SHARPE: "The inner force of a moving electron."

(17) Dr. W. H. ROEVER: "Brilliant points of curves and surfaces."

(18) Mr. FRANK IRWIN: "Transformations of the elements $x, y, y', \cdots, y^{(k)}$ that carry a union of such elements over into a union."

Mr. Irwin's paper was communicated to the Society through Professor Bouton. In the absence of the authors, Professor Haskins's paper was read by Professor Mason, Dr. Sharpe's second paper was read by Professor Snyder, and Dr. Sharpe's first paper and the papers of Professor Carmichael, Professor

Van Vleck, Professor Dickson, Dr. Hill, Professor Miller, Dr. Roever, and Mr. Irwin were read by title.

Professor Van Vleck's paper will appear in the April *Transactions*. The papers of Professor Dickson, Professor Mason, and Dr. Hill appeared in the April BULLETIN. Professor Young's second paper and Dr. Sharpe's second paper appear in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael obtains several propositions concerning the numerical factors of arithmetic forms determined thus: Let $Q_n(x) = 0$ be the equation whose roots are the primitive nth roots of unity. In $Q_n(x)$ put $x = \alpha/\beta$, α and β being relatively prime positive integers and $\alpha > \beta$. Multiply this expression by β raised to a power equal to the degree of $Q_n(x)$. A study is made of the factors of the resulting forms. Several propositions are proved which are generalizations of those obtained by Dickson in an article "On the cyclotomic function" (*American Mathematical Monthly*, volume 12, pages 86–89).

2. The series which Professor Carmichael studies in this paper, with the remainder term which he finds, is as follows:

$$
\begin{aligned}
f(a+x) = f(a) + \frac{x}{m}\Big[f'\left(a+\frac{x}{m}\right) + f'\left(a+\frac{2x}{m}\right) + \cdots \\
+ f'\left(a + \frac{m-1}{m}x\right) + f'(a+x)\Big] - \frac{x}{m\cdot 2}\left[f'(a+x) - f'(a)\right] \\
- \frac{B_1 x^2}{m^2\cdot 2!}\left[f''(a+x) - f''(a)\right] + \frac{B_2 x^4}{m^4\cdot 4!}\left[f^{\mathrm{IV}}(a+x) - f^{\mathrm{IV}}(a)\right] \\
- \cdots + (-1)^n \frac{B_n x^{2n}}{m^{2n}\cdot(2n)!}\left[f^{(2n)}(a+x) - f^{(2n)}(a)\right]. \\
+ x^{2n+2}\Big[\frac{1}{(2n+2)!} + \frac{1}{2m(2n+1)!} + \frac{B_1}{m^2\cdot 2!(2n)!} \\
+ \frac{B_2}{m^4\cdot 4!(2n-2)!} + \cdots + \frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)!\,4!} + \frac{B_n}{m^{2n}\cdot(2n)!\,2!}\Big] \\
\times \left[f^{(2n+2)}(a+\phi_1 x) - f^{(2n+2)}(a+\phi_2 x)\right],
\end{aligned}
$$

B_1, B_2, $\cdots$ being Bernoulli's numbers, and ϕ_1 and $\phi_2 < 1$ and > 0.

This series, without remainder term, was given by S. A. Corey in the *Annals of Mathematics*, second series, volume 5, number 4 (July, 1904).

3. The field due to a moving charge has been discussed by many writers : Langevin in particular has given an elegant investigation of the field due to a moving electron. Poincaré has partially solved Langevin's problem by the use of the Lorentzian transformation. In Dr. Sharpe's paper the Lorentzian transformation is applied to the definite integral expressions for the scalar and vector potentials of the field instead of to the differential equations of the field. By aid of the transformation the problem of Langevin is reduced to the simpler problem where the velocity of the electron vanishes at the instant considered. By transforming back again, the solution of Langevin's problem is obtained with a new form for the magnetic force.

4. In Professor Snyder's paper, curves of genus 6 are defined as the partial intersection of five particular quadric spreads in space of five dimensions, R_5, and one arbitrary spread. By proper projection, this curve of order 10 can be reduced to a plane sextic with four double points. The coordinate spaces become adjoint cubics, in terms of which an equation of any sextic of this type can be written. Every possible birational transformation of this curve into itself can be generated by four linear operators of order two. The most interesting groups are G_{120}, to which a sextic with four double points at the vertices of a quadrangle belongs, a G_{150} leaving invariant a non-singular quintic (not reducible to a sextic), and a cyclic G_{26} of a certain hyperelliptic curve. The study of the non-singular quintic from this standpoint involves that of the Veronese surface in R_5.*

5. A chain in the ordinary complex projective geometry may be defined analytically as any class of points projective with the real points of a line, from which follows that there is one and only one chain through three given points of a line. With the usual representation of complex numbers on a sphere a

* Wiman, *Bihang till k. Sven. Vet. Akad. Handlingar*, vol. 21 (1895), two articles. Clebsch, Geometrie, vol. 1, p. 695 ff. Segre, *Ann. di mat.* (2), vol. 22 (1894).

chain is simply a circle. A chain may also be defined directly in terms of three points on a line by means of the notion of a net of rationality on a line and order relations, as was done in a paper presented to the Society by Professors Veblen and Young at its last meeting. In his first paper Professor Young develops some of the fundamental properties of the chains of a line by purely synthetic methods, the results being well known in the theory of functions of a complex variable when the chains are identified with the circles on the sphere of the complex variable, or the Argand plane. The proofs are in many cases much simpler than the usual analytic proofs. Moreover it appears important to emphasize the fact that the function-theoretic theorems in question form an elementary chapter in projective geometry.

The system of chains through two given points M, N is called the system $T(M, N)$. Through any point P distinct from M, N there is one and only one chain cutting each of the chains through M, N in points harmonic with M, N. Every such chain is said to be "about" M, N; the system of all chains about M, N is denoted by $A(M, N)$. The two systems T and A are fundamental in the classification of projectivities with distinct double points. If a chain is about two points M, N, these points are said to be conjugate with respect to the chain, and one is the inverse of the other. Then if every point on a chain is defined as its own inverse, the existence and uniqueness of the inverse of any point of a line with reference to a given chain follows. Two chains are defined as orthogonal if one contains two points conjugate with respect to the other. The projectivities on a line are then classified, with reference to their invariant figures, into involutoric, elliptic, hyperbolic, loxodromic, and parabolic projectivities.

11. In the paper of Professor Bliss a method of deriving Euler's equation in the calculus of variations by means of the theory of invariant integrals of the form

$$\int \left\{ A(x, y) + B(x, y)\frac{dy}{dx} \right\} dx$$

is exhibited. The method permits one to derive very simply the result of Du Bois-Reymond and Hilbert, that for a regular problem the minimizing curve can not have corner points where the slope is discontinuous. A considerable simplification in the presentation of the whole theory of the simplest problem in the plane is also effected, because the notion of an invariant integral is one which can be frequently applied in the derivation of the further conditions.

12. The second law of the mean is fundamental in the classical Dirichlet discussion of Fourier's series. The fundamental theorem of Fourier's constants, however, as proved by de la Valleé Poussin and Hurwitz, does not involve the second law of the mean. In Professor Haskins's note it is shown that the second law of the mean is a simple consequence of the fundamental theorem of Fourier's constants.

13. In the *Leipziger Berichte* for 1889 Lie studied those infinitesimal contact transformations whose characteristic function is of the form $\Omega(x, y)\sqrt{1 + p^2}$ and indicated their importance in dynamics. In the present note Professor Kasner shows that the alternant of any two transformations of this type is necessarily a point transformation. The result holds for any number of variables and may be applied to many allied theories.

14. In connection with any surface S we may consider the curves cut out by the ∞^3 planes of space. By projecting these orthogonally or centrally upon a fixed plane we obtain the triply infinite systems which are investigated in Professor Kasner's paper. Some of the properties derived are analogous to those which hold for dynamical trajectories: in particular the focal locus is circular for both types. The general classes are however distinct, and overlap only when the surface S is cylindrical or conical. The developable surfaces give rise to especially simple results. The theory worked out belongs to general projective geometry.

15. The group G formed by the $\phi(n)$ numbers which are less than n and prime to n, when they are combined with respect to multiplication modulo n, has been called by H. Weber the most important example of an abelian group of finite order. The object of Professor Miller's note is to point out how several fundamental theorems relating to the periods of decimal fractions may readily be proved by means of this group. Assuming $n > 1$ and prime to 10, it results that 10, or

its residue modulo n, is a number in G and that the powers of 10 generate a cyclic subgroup K whose order k is the length of the period of $1/n$, since K includes 1 and multiplying a decimal fraction by 10 simply moves its decimal point one place to the right. The operators of K have therefore the characteristic property that they simply affect the position of the decimal point of any number divided by n with which they are combined under G, moving it respectively $1, 2, \cdots, k$ places to the right. If these operators are applied a second time in order they will move this point respectively $k + 1$, $k + 2$, $\cdots$, $2k$ places to the right. As the k distinct numbers modulo n obtained in the second case are the same as those obtained in the first, and each of the $\phi(n)$ numbers divided by n gives rise to a different decimal fraction it results that k is the length of the period of *every* number of G divided by n.

From the fact that G always involves -1 it follows that the numbers of G may be divided into pairs differing only with respect to sign, and hence the mantissa of the one divided by n may be obtained by subtracting from g each digit in the mantissa of the other divided by n. If K involves -1, the mantissas of the two numbers differing only in sign divided by n may be obtained from each other by a cyclic permutation and hence the entire period of such a number may be obtained by subtracting from g each digit of half the period. When the index of K under G is odd, in particular when 10 is a primitive root of n, K must involve -1 and hence half the period of every number of G divided by n may be obtained by subtracting from g the digits of the other half in order. The index of K under G is the number of distinct periods in the fractions obtained by dividing by n all the numbers which are prime to n. If the mantissa of a number of G divided by n were composed of g's, that of the other number of the pair would be composed of o's; *i. e.*, the otherwise evident fact that g cannot constitute the period of a proper fraction in decimal form may be regarded as a special case of the given theorems. With the exception of the periods o and g, any arbitrary set of numbers may evidently be the period of a proper fraction.

17. In this paper, which is an extension of one reported in the BULLETIN (series 2, volume 8 (1901–02), page 279) and published in the *Annals of Mathematics* (series 2, volume 3 (1901–02), page 113), Dr. Roever begins with a description of

the optical phenomenon which suggested the problems considered. A brilliant point of a curve c with respect to two congruences is defined as a point P of c at which the curves of the two congruences which pass through P make equal angles with c at P. A system of three congruences is then considered, and the condition found that a point P should be a brilliant point of the curve of one of these which passes through P, with respect to the other two. The condition consists in the vanishing of a certain expression Ω, which involves the functions that appear in the systems of differential equations of the congruences. It is found that the expression Ω possesses a certain kind of invariance. Thus, if each of the three congruences represented in Ω be replaced by a certain congruence involving an arbitrary function, the expression Ω formed from the new congruences differs from that formed from the old ones by a non-vanishing factor. Therefore the equation $\Omega = 0$ is the locus of the brilliant points of each of an infinite number of congruences with respect to each of an infinite number of pairs of congruences. Two special kinds of brilliant point are considered in great detail. In terms of either of these, the brilliant point of a surface with respect to a pair of congruences, may be defined by considering a congruence normal to the surface. It is shown that under certain restrictions brilliant points of both curves and surfaces, may be regarded as points of contact. It is also shown that optical interpretations of one or another kind may be given to most of the results. Numerous examples illustrate the results.

18. Mr. Irwin shows that the only transformations of the elements $(x, y, y', \cdots, y^{(k)})$ carrying a union of these elements over into a union are extended contact transformations.

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Secretary.