ON CERTAIN CONSTANTS ANALOGOUS TO FOURIER'S CONSTANTS.

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IN the course of an article which appeared recently in the *Rendiconti del Circolo Matematico di Palermo*,* Landau has reproduced two proofs of the following theorem:

A. If the function $\psi(x)$ *is continuous in the interval* $(0 \leqq x \leqq 1)$ *and if*

$$(1) \qquad \int_0^1 x^\nu \psi(x) dx = 0 \qquad (\nu = 0, 1, 2, \cdots),$$

then

$$\psi(x) = 0 \qquad (0 \leqq x \leqq 1).$$

The proofs that Landau gives in detail are due to Lerch and Stieltjes. In addition he cites a second proof due to Stieltjes and a proof due to Phragmen.

As far as I am able to learn, no one seems to have mentioned the fact that this theorem, of which so many proofs have been given, is essentially equivalent to a theorem due to Hurwitz† which may be stated as follows:

B. If in the interval $(0 \leqq x \leqq 2\pi)$ *the function* $f(x)$ *is finite and integrable and if all of its Fourier's constants are zero, then* $f(x)$ *is zero at every point of the interval at which it is continuous.*

Theorem (A) may be deduced from (B) as follows:

It is obvious that if $\psi(x)$ is finite and integrable in the interval $(0 \leqq x \leqq 1)$ and if condition (1) is fulfilled, then the function

$$f(y) = \psi(y/2\pi)$$

satisfies all the conditions of Hurwitz's theorem. For

* Vol. 25 (1908), p. 1.

† Cf. *Mathematische Annalen*, vol. 57 (1903), p. 440. Cf. also Bonnet, *Mémoires de l'Académie de Belgique*, vol. 23 (1850), p. 11.

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\cos nydy = 2\int_0^1 \psi(x)\cos 2n\pi xdx$$
$$= 2\left[\int_0^1 \psi(x)dx - \frac{4n^2\pi^2}{2\,!}\int_0^1 x^2\psi(x)dx + \cdots\right] = 0\,*,$$

and similarly

$$\frac{1}{\pi}\int_0^{2\pi} f(y)\sin nydy = 0.$$

Consequently $f(y)$ is zero at every point of the interval $(0 \leqq y \leqq 2\pi)$ at which it is continuous, and hence $\psi(x)$ is zero at every point of the interval $(0 \leqq x \leqq 1)$ at which it is continuous.

Thus Hurwitz's theorem gives us an immediate proof of theorem (A) in the more general case in which $\psi(x)$ is merely finite and integrable. However, as Professor Bôcher has pointed out,† Hurwitz's theorem holds even when $f(x)$ becomes infinite at a finite number of points, provided

$$\int_0^{2\pi} |f(x)|\,dx$$

converges. Therefore theorem (A) holds for the case in which

$$\int_0^1 |\psi(x)|\,dx$$

is convergent.‡

I will now give a rather simple proof of theorem (A) for a still more general case and will then obtain theorem (B) under equally general conditions, as a consequence of this theorem. As far as I know, neither theorem has been proved before with the same degree of generality.

I will begin by proving two lemmas.

LEMMA 1. *If* $\psi(x)$ *is continuous in the interval* $(0 \leqq a \leqq x \leqq b)$ *and if*

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* We have a right to multiply the series for the cosine by a function that is finite and integrable and to integrate it term by term since the series is uniformly convergent throughout any finite interval.

† Cf. *Annals of Mathematics*, vol. 7 (1906), p. 101.

‡ This is in view of the fact that we have a right to multiply the series for the cosine by a function that is absolutely integrable and to integrate term by term.

$$\int_a^b x^{\gamma+n}\psi(x)dx = 0 \quad (\gamma \geqq 0\,;\ n = 0, 1, 2, \cdots), \tag{2}$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

Since $\psi(x)$ is continuous, we know from a theorem due to Weierstrass * that we can develop it in a uniformly convergent series of polynomials in x so that we have

$$\psi(x) = P_0(x) + P_1(x) + P_2(x) + \cdots. \tag{3}$$

We now multiply both sides of equation (3) by $x^\gamma\psi(x)$, and integrate from a to b. Since the series on the right hand side of (3) is uniformly convergent, we have a right to integrate term by term and hence in view of condition (2) the whole right hand side vanishes, so that we get

$$\int_a^b x^\gamma[\psi(x)]^2dx = 0.$$

Consequently, since $\psi(x)$ is continuous, it must be zero at every point of the interval $(a \leqq x \leqq b)$ and the lemma is proved.

By a change of variable this lemma can be thrown into the slightly more general form :

LEMMA 2. *If $\psi(x)$ is continuous in the interval $(0 \leqq a \leqq x \leqq b)$ and if*

$$\int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0 \quad (\gamma \geqq 0, \alpha > 0\,;\ n = 0, 1, 2, \cdots), \tag{4}$$

then

$$\psi(x) = 0 \qquad (a \leqq x \leqq b).$$

We are now in a position to prove the following two theorems :

THEOREM I. *If in the interval $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ is finite save for a finite number of points, and is integrable, and if furthermore*

$$\int_a^b x^{\gamma+n\alpha}\psi(x)dx = 0\,\dagger \quad (\gamma \geqq 0, \alpha > 0\,;\ n = 0, 1, 2, \cdots), \tag{5}$$

* Cf. Picard, Traité d'Analyse, vol. 1, 2d ed., p. 279. Lerch's proof referred to above is based on a slightly different form of this theorem. The proof as I have given it, however, is considerably briefer.

† Condition (5) is not essentially more general than condition (1), since it involves merely a change of notation. The real generalization obtained in Theorem I is in the removal of restrictions upon $\psi(x)$.

then $\psi(x)$ *is zero at every point of the interval* $(a \leqq x \leqq b)$ *at which it is continuous.*

Let

$$\int_a^x x^\gamma \psi(x) dx = \chi(x). \tag{6}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$ and in view of (6) and (5)

$$\chi(a) = 0, \quad \chi(b) = 0. \tag{7}$$

Let us take the integer m so great that $m\alpha > 1$. Then for $n = m + k \geqq m$ we have from an integration by parts

$$\int_a^b x^{\gamma + n\alpha} \chi(x) dx = [x^{n\alpha}\chi(x)]_a^b - \int_a^b x^{m\alpha - 1 + k\alpha} \chi(x) dx \quad (k=0, 1, 2, \cdots)$$

or in view of (5) and (7)

$$\int_a^b x^{m\alpha - 1 + k\alpha} \chi(x) dx = 0 \qquad (k = 0, 1, 2, \cdots)$$

and hence from Lemma 2

$$\chi(x) = \int_a^x x^\gamma \psi(x) dx = 0 \qquad (a \leqq x \leqq b). \tag{8}$$

Differentiating (8), we see that $x^\gamma \psi(x)$, and consequently $\psi(x)$, is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous.*

THEOREM II. *If in the interval* $(0 \leqq a \leqq x \leqq b)$ $\psi(x)$ *is finite save for a finite number of points, and is integrable, and if furthermore*

$$\begin{aligned} &\int_a^q \psi(x) \cos nx dx = 0 \qquad (n = 0, 1, 2, \cdots), \\ &\int_a^b \psi(x) \sin nx dx = 0 \qquad (n = 1, 2, \cdots), \end{aligned} \tag{9}$$

then $\psi(x)$ *is zero at every point of the interval* $(a \leqq x \leqq b)$ *at which it is continuous.*

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* If $a = 0$ and $\gamma > 0$ it does not necessarily follow that, when $x^\gamma \psi(x)$ is zero for $x = 0$, $\psi(x)$ is zero also. However, when (8) is fulfilled for $a = 0$, it is easily seen from the integrability of $\psi(x)$ that it must be zero for $x = 0$ if it is continuous there.

We know that in any fixed interval x^n can be developed in a Fourier's series that is uniformly convergent throughout the interval and which when differentiated term by term will yield a series that is uniformly convergent throughout the same interval and which represents there the derivative of x^n.* Hence we have

$$x^n = a_0 + a_1 \cos x + b_1 \sin x + \cdots \quad (a \leqq x \leqq b), \tag{10}$$

$$nx^{n-1} = - a_1 \sin x + b_2 \cos x - \cdots \quad (a \leqq x \leqq b). \tag{11}$$

In view of (10) and the hypothesis that $\psi(x)$ is integrable in the interval $a \leqq x \leqq b$, we have

$$\begin{aligned} b^n \int_a^b \psi(x)dx = a_0 \int_a^b \psi(x)dx + a_1 \cos b \int_a^b \psi(x)dx \\ + b_1 \sin b \int_a^b \psi(x)dx + \cdots. \end{aligned} \tag{12}$$

Now let

$$\int_a^x \psi(x)dx = \chi(x). \tag{13}$$

Then $\chi(x)$ is continuous in the interval $(a \leqq x \leqq b)$, and since the series on the right hand side of (11) is uniformly convergent, we have

$$\begin{aligned} - n \int_a^b x^{n-1}\chi(x)dx = a_1 \int_a^b \chi(x) \sin xdx \\ - b_1 \int_a^b \chi(x) \cos xdx + \cdots. \end{aligned} \tag{14}$$

Adding (12) and (14), we get

$$\begin{aligned} & b^n \int_a^b \psi(x)dx - n \int_a^b x^{n-1}\chi(x)dx \\ & \quad = a_0 \int_a^b \psi(x)dx + a_1 \left[\cos b \int_a^b \psi(x)dx + \int_a^b \chi(x) \sin xdx \right] \\ & \qquad + b_1 \left[\sin b \int_a^b \psi(x)dx - \int_a^b \chi(x) \cos xdx \right] + \cdots \end{aligned}$$

or

* For sufficient conditions that a function may be developed in a uniformly convergent Fourier's series and that a uniformly convergent Fourier's series representing its derivative may be obtained by differentiating that series term by term see Professor Bôcher's article referred to above.

$$\int_a^b x^n\psi(x)dx = a_0\int_a^b \psi(x)dx + a_1\int_a^b \psi(x)\cos x dx$$
$$+ b_1\int_a^b \psi(x)\sin x dx + \cdots = 0 \quad (n = 1, 2, \cdots).$$

Hence $\psi(x)$ satisfies the conditions of Theorem I, and therefore it is zero at every point at which it is continuous.

The method of proof used in Theorem II may be applied in the case of developments in terms of any other normal functions, such as Bessel functions, Legendre's polynomials, etc., whenever we know that x^n can be developed in a convergent series of such functions which, when differentiated term by term, will yield a uniformly convergent series that represents the derivative of x^n.* The method will enable us to show in such cases that if the coefficients of the development corresponding to any function which we know to be finite, save for a finite number of points, and integrable, are all zero, the function is zero at every point at which it is continuous.

NOTE ON THE SECOND VARIATION IN AN ISOPERIMETRIC PROBLEM.

BY DR. ELIJAH SWIFT.

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SUPPOSE we have before us the simplest type of isoperimetric problem, namely to determine x and y as functions of a parameter t, so that the definite integral

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt$$

shall be a minimum, while another definite integral

$$K = \int_{t_0}^{t_1} G(x, y, x', y')dt$$

* The existence of such developments can be proved for some of these cases by means of some theorems discussed by Stekloff. Cf. *Mémoires de l'Académie de St. Pétersbourg*, ser. 8, vol. 15 (1904).