CRITERIA FOR THE IRRREDUCIBILITY OF A RECIPROCAL EQUATION.

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1. A reciprocal equation \( f(x) = x^m + \cdots = 0 \) is one for which

\[ x^m f(1/x) \equiv cf(x). \]

Replacing \( x \) by \( 1/x \), we see that \( f \equiv c_2 f, c = \pm 1 \). Now \( f(x) \) has the factor \( x \pm 1 \) and hence is reducible, unless \( m \) is even and \( c = +1 \). Further discussion may therefore be limited to equations

\[ F(x) = x^{2n} + c_1 x^{2n-2} + \cdots + c_2 x^2 + c_1 x + 1 = 0 \]

of even degree and having

\[ x^{2n} F(1/x) \equiv F(x). \]

Let \( R \) be a domain of rationality containing the \( c \)'s.

Under the substitution

\[ x + 1/x = y, \]

\( x^{-n} F(x) \) becomes a polynomial in \( y \),

\[ \phi(y) = y^n + k_1 y^{n-1} + \cdots + k_n, \]

with coefficients in \( R \). By a suitable choice of the \( c \)'s, the \( k \)'s may be made equal to any assigned values.

We shall establish in §§ 2–7 the following:

**THEOREM.** Necessary and sufficient conditions for the irreducibility of \( F(x) \) in the domain \( R \) are

(I) \( \phi(y) \) must be irreducible in \( R \).

(II) \( F(x) \) must not equal a product of two distinct irreducible functions of degree \( n \).

The second condition is discussed in §§ 8–10.

2. The irreducibility of \( F(x) \) in \( R \) implies that of \( \phi(y) \).

For, if
\( \phi(y) = (y' + \cdots)(y^n + \cdots), \quad (l + m = n, \quad l > 0, \quad m > 0), \)

then would

\[ F(x) = x'[((x + 1/x)' + \cdots) \cdot x^m((x + 1/x)^m + \cdots)]. \]

3. If \( F(x) \) has in \( R \) an irreducible factor

\[ A(x) = x^{2r+1} + a_1x^{2r} + \cdots + a_{2r}x + a_{2r+1} \]

of odd degree, then \( F(x) \) has the irreducible factor.

\[ B(x) = \frac{x^{2r+1}}{a_{2r+1}} A \left( \frac{1}{x} \right) = x^{2r+1} + \frac{a_{2r}}{a_{2r+1}} x^{2r} + \cdots + \frac{a_1}{a_{2r+1}} x + \frac{1}{a_{2r+1}}, \]

not identical with \( A(x) \). For, from \( F \equiv A Q \) and (2) follows

\[ F(x) \equiv B(x) Q', \quad Q' \equiv a_{2r+1} x^{2n-2r-1} Q(1/x). \]

Next, if \( B \equiv A \), then

\[ A = x^{2r+1} \pm 1 + a_1 x^{2r-1} \pm 1 + \cdots + a_r x^{r(\pm 1)}, \]

so that \( A \) would have the factor \( x \pm 1 \) and be reducible.

4. If \( \phi(y) \) is irreducible in \( R \), \( F(x) \) has in \( R \) no irreducible factor \( A \) of odd degree \( < n \). For, if so, \( P \equiv AB \), where \( B \) is given in § 3, would be a self-reciprocal factor of \( F(x) \). In fact,

\[ P(1/x) = A(1/x) A(x)/x^{2r+1} a_{2r+1} , \quad x^{2(2r+1)} P(1/x) = AB = P(x). \]

Hence, in view of (3), \( x^{-(2r+1)} P(x) \) would equal a factor of degree \( 2r + 1 \) of \( \phi(y) \).

5. If \( F(x) \) has in \( R \) an irreducible factor

\[ A(x) = x^{2r} + a_1 x^{2r-1} + \cdots + a_{2r-1} x + a_{2r} \]

of even degree, then \( F(x) \) has the irreducible factor

\[ B(x) = \frac{1}{a_{2r}} x^{2r} A \left( \frac{1}{x} \right). \]

If \( B(x) \equiv A(x) \), \( A \) is self-reciprocal, viz.,

\[ A(x) = x^{2r} + 1 + a_1(x^{2r-1} + x) + \cdots + a_{2r-1}(x^{r+1} + x^{r-1}) + a_r x^r. \]
In fact, the conditions for \( B = A \) are
\[
a_{2r} = \pm 1, \quad a_{2r-1} = \pm a_1, \quad a_{2r-2} = \pm a_2, \quad \cdots.
\]
For the lower signs, \( A \) has the factor \( x^2 - 1 \), contrary to hypothesis.

6. If \( \phi(y) \) is irreducible in \( R \), \( F(x) \) has in \( R \) no irreducible factor \( A \) of even degree \( < n \). For, by § 5, either \( B \) is distinct from \( A \) so that \( AB \) is a self-reciprocal factor of \( F(x) \), or else \( A \) itself is a self-reciprocal factor. In either case \( \phi(y) \) would have in \( R \) a factor of degree \( < n \).

7. It follows from §§ 4, 6 that, when \( \phi(y) \) is irreducible in \( R \), \( F(x) \) has no irreducible factor of degree \( < n \). Further, by §§ 3, 5, an irreducible factor \( A(x) \) of degree \( n \) implies a second irreducible factor \( x^n A(1/x) \), algebraically distinct from \( A(x) \). The theorem of § 1 is therefore proved.

8. It remains to consider the case \( F = AB \),
\[
A = x^n + a_1 x^{n-1} + \cdots + a_n,
\]
\[
B = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + \frac{1}{a_n},
\]
where \( A \) and \( B \) are distinct irreducible functions in \( R \). To determine the \( a_0 \) we have \( n \) distinct relations
\[
\begin{align*}
(5) \quad & a_1 + a_{n-1}/a_n = c_1, \\
& a_2 + (a_1 a_{n-1} + a_{n-2})/a_n = c_2, \\
& \quad \vdots
\end{align*}
\]
We may eliminate \( a_1, \ldots, a_{n-1} \) and obtain an equation for \( a_n \).
As shown in § 9, this equation is of degree \( 2^n \). Except for certain sets of values of the \( c_i \), we may express \( a_1, \ldots, a_{n-1} \) rationally in terms of \( a_n \); the problem is then reduced to the consideration of the rationality of a root of the equation of degree \( 2^n \). This equation for \( a_n \) is a reciprocal equation. In fact, if we set
\[
A_1 = a_{n-1}/a_n, \quad \cdots, \quad A_{n-1} = a_1/a_n, \quad A_n = 1/a_n,
\]
equations (5) become
\[
\begin{align*}
(5') & A_1 + A_{n-1}/A_n = c_1, \\
& A_2 + (A_1 A_{n-1} + A_{n-2})/A_n = c_2, \\
& \quad \vdots
\end{align*}
\]
That the equations (5') are throughout of the same form as equations (5) is evident from the fact that we have merely interchanged the rôles of the factors \( A \) and \( B \) of \( F \). Hence the equation in \( a_n \), obtained by eliminating \( a_1, \ldots, a_{n-1} \) from
(5), is identical with the equation in $A_n = 1/a_n$, obtained from (5').

9. Denote the roots of $F = 0$ by

$$a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots, a_n, a_n^{-1}. $$

A factorization $F' = AB$, of the kind considered in § 8, corresponds uniquely to a separation of the roots (6) into two sets each of $n$ roots, such that reciprocal roots belong to different sets. Hence the roots of the first set may be selected in

$$\frac{2n(2n - 2)(2n - 4) \cdots 2}{n!} = 2^n$$

ways. The number of factors $A$ is thus $2^n$.

10. For $n = 2$, we set $a_1 = \alpha$, $a_2 = \beta$, and have

$$a + \beta + \alpha^{-1} + \beta^{-1} = c_1, 2 + \alpha \beta^{-1} + \beta \alpha^{-1} + \alpha \beta + \alpha^{-1} \beta^{-1} = c_2. $$

From these we derive

$$\alpha^2 + \beta^2 + \alpha^{-2} + \beta^{-2} = c_1^2 - 2c_2. $$

Hence $\alpha \beta + \alpha^{-1} \beta^{-1}$ and $\alpha \beta^{-1} + \alpha^{-1} \beta$ are the roots of

$$z^2 - (c_2 - 2)z + c_1^2 - 2c_2 = 0. $$

The quartic for $a_2(§8)$ is obtained by setting

$$z = a_2 + a_2^{-1}. $$

By (7), $\alpha + \beta$ is a rational function of $\alpha \beta$ and $c_1$ when $c_1 \neq 0$. Hence, for $c_1 = 0$, the necessary and sufficient conditions for the factorization $F' = AB$ in $R$ are that the roots

$$z_n = \frac{1}{2}(c_2 - 2) \pm [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{1/2}$$

of (8) be rational and that one of the values $(z_n^2 - 4)^{1/2}$ be rational, so that (9) shall lead to a rational value of $a_2$. Incorporating the condition that (4) shall be irreducible in $R$, we obtain the

**Theorem.* The necessary and sufficient conditions that

$$a^4 + c_1 a^3 + c_2 a^2 + c_3 a + 1 \ (c_1 \neq 0)$$

* For other proofs by the writer, see *Amer. Math. Monthly*, vol. 10 (1903), p. 221; vol. 15 (1908), p. 75. The first paper cited also treats reciprocal sextic equations.
shall be irreducible in a domain R are that \((c_1^2 - 4c_2 + 8)^{1/2}\) be irrational, and that either \(l = [(1 + \frac{1}{2}c_2)^2 - c_2^2]^{1/2}\) be irrational or else \(l\) rational and \([\frac{1}{2}c_2^2 - c_2^4 - 2 \pm (c_2 - 2)l]\) both irrational.

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

\[
x' = \pm \frac{\alpha x + \beta}{\beta x + \alpha} \quad (\alpha^2 \neq \beta^2).
\]

Then \(y\), given by (3), undergoes the transformation

\[
y' = \pm \frac{(\alpha^2 + \beta^2)y + 4\alpha\beta}{\alpha\beta y + \alpha^2 + \beta^2}.
\]

The transformation on \(\frac{1}{2}y\) is the square of (11).

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A NEW GRAPHICAL METHOD FOR QUATERNIONS.

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1. Any quaternion \(q\) may be written in the form

\(q = (w + xi) + (y + zi)j\). For convenience let us represent numbers of the form \(w + xi\) (practically equivalent to ordinary complex numbers save in their products by \(j\)) by Greek characters, so that \(q\) may be written

\(q = \alpha + \beta j\),

where for any number \(\beta\) we have \(\beta j = j\bar{\beta}\), \(\bar{\beta}\) being the conjugate of \(\beta\).

The tensor of \(q\) is then the square root of the sum of the squares of the moduli of \(\alpha, \beta\). Also the scalar of \(q\) is \(\frac{1}{2}(\alpha + \bar{\alpha})\), that is, the real part of \(\alpha\).

2. The product of \(q = \alpha + \beta j\) and \(r = \gamma + \delta j\) is

\(qr = (\alpha\gamma - \bar{\beta}\delta) + (\alpha\delta + \beta\gamma)j\),

and also we have

\(rq = (\alpha\gamma - \beta\delta) + (\bar{\alpha}\delta + \beta\gamma)j\).