V. *Conclusion.*

The method used to set up the convergency conditions for the infinite series expressing the roots of the four-term equation derived from the equations formed in accordance with (a), (b) (c) of Article I when the convergency conditions of the three-term equation are known, and to set up the convergency conditions for the five-term equation when the convergency conditions for the four-term equation are known, can be used to set up the convergency conditions for the t-term equation when the convergency conditions for the $(t-1)$-term equation are known.

In fact, the convergency conditions for an equation of any number of terms can be written mechanically.

For the t-term equation

$$f(y) = 0 \tag{57}$$

it is always possible to determine the s of the substitution

$$y = z^s \tag{58}$$

so that the convergency conditions of the infinite series expressing the roots of the t-term equation

$$f(z^s) = 0 \tag{59}$$

derived from the equation formed from (59) in accordance with (a) of Article I, or derived from the two equations formed from (59) by (b) and (c) of Article I are satisfied. The roots of the t-term equation (57) are found from the roots of the equation (59) by substituting in (58).

It is therefore always possible to find all the roots of an algebraic equation by means of infinite series.

LEHIGH UNIVERSITY,
April, 1908.

THE DEDUCTION OF THE ELECTROSTATIC EQUATIONS BY THE CALCULUS OF VARIATIONS.

BY DR. ARTHUR C. LUNN.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

THE construction of a mathematical theory of classes of physical phenomena for which no detailed mechanical explanation is available has in several instances led to the enunciation of some general minimal property, usually relating to the distribution of energy and analogous to the principles of least action and of virtual velocities in dynamics. Such a variational "principle" may be regarded either as a concise equivalent for a system of algebraic or differential equations already obtained, or as a form of hypothesis from which those equations may be deduced. The systematic use in the latter sense by Lagrange of the principle of virtual velocities has given his Mécanique Analytique a unity and simplicity far beyond those of most later writings on dynamics.

Corresponding to the equations of electrostatic equilibrium, a number of related forms have been given of a minimizing condition on the energy. These differ among themselves mainly in the character of the variations implied, whether for example variations of a field in which there are charges given or of the charges in a given field.* But in all the forms known to the writer the existence of the electrostatic potential is assumed, while from the point of view of the theory of vector fields it seems more fitting to avoid this assumption if possible, since it is only a special kind of vector field which has a scalar potential. The following deduction according to the formal methods of the calculus of variations, showing how the potential function may appear simply as a lagrangian multiplier, is analogous to the method used by Lagrange for the theory of incompressible fluids, in which the hydrostatic pressure is not introduced at the start as a physical concept, but appears as the multiplier corresponding to the differential condition of invariable volume.†

Adopting the point of view of Faraday and Maxwell, let it be supposed that the electrostatic phenomena can be completely described in terms of a vector point function $\mathbf{E}$, the electric force on unit charge, and a scalar point function ϵ, the dielectric coefficient;‡ that in terms of these the displacement and the volume and surface densities of charge are

$$(1) \qquad \mathbf{D} = \epsilon \mathbf{E}, \quad \rho = \operatorname{div} \mathbf{D}, \quad \sigma = \mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'',$$

where $\mathbf{n}'$, $\mathbf{n}''$ are opposite unit normals of a surface on which the

* See for example: Webster, The Theory of Electricity and Magnetism, § 131; Weber, Differentialgleichungen der mathematischen Physik, I, p. 310.

† Mécanique Analytique, part I, sec. VII.

‡ The vector notation and units are those of Heaviside, Electromagnetic Theory.

surface density is the discontinuity in the normal component of **D**; and that the total energy stored in a given volume by means of the electrostatic strain is

$$W = \tfrac{1}{2}\int \mathbf{E}\mathbf{D}dv. \tag{2}$$

If the total charge on a conductor be supposed to include the surface charge on the boundary between conductor and dielectric, then the charge belonging to any conductor with index i is

$$Q_i = \int \operatorname{div} \mathbf{D} dv_i + \int (\mathbf{n}'\mathbf{D}' + \mathbf{n}''\mathbf{D}'')dS_i,$$

but by transformation of the volume integral to a surface integral this becomes

$$Q_i = \int \mathbf{n}'\mathbf{D}'dS_i, \tag{3}$$

where $\mathbf{n}'$ is the unit normal into the dielectric. If **D** should have a discontinuity at a surface within the conductor, the terms in the surface integrals belonging to opposite sides of that surface would be cancelled by corresponding terms from the volume integral, so that the integration need cover only the bounding surface. It is simpler, however, to assume that ϵ and therefore **D** are zero everywhere within the conductor, so that the only charge is on the bounding surface, in which case formula (3) still holds.

If ϵ be supposed to vanish within the substance of a conductor and to be a definite physical coefficient different from zero at each point of the dielectric, then the form here to be considered of the condition of equilibrium is that the first variation of W shall vanish for all variations of **E** consistent with giving assigned values to the volume and surface densities in the dielectric and to the total charge on each conductor; or, analytically, that with the integration extended throughout the dielectric

$$\delta\int \epsilon \mathbf{E}^2 dv = 0 \tag{4}$$

for all variations $\delta\mathbf{E}$ satisfying the conditions

$$div(\epsilon\mathbf{E}) = \rho, \tag{I}$$

where ρ is an assigned function throughout the volume of the dielectric; and

(II) $$\epsilon'\mathbf{E}'\mathbf{n}' + \epsilon''\mathbf{E}''\mathbf{n}'' = \sigma,$$

where σ is an assigned function over all surfaces of discontinuity in the dielectric; and

(III) $$\int \epsilon'\mathbf{E}'\mathbf{n}' dS_i = Q_i,$$

where each Q_i is an assigned constant for the corresponding conductor.

To this conditioned variation of W there corresponds according to Lagrange's method of multipliers a free variation of the modified integral

(5) $$U = W - \int V \operatorname{div}(\epsilon\mathbf{E})dv - \sum V_i \int \epsilon'\mathbf{E}'\mathbf{n}' dS_i,$$

where the undetermined function V is the multiplier corresponding to the differential condition (I) and the constants V_i are the multipliers corresponding to the integral conditions (III). But

$$\int V \operatorname{div}(\epsilon\mathbf{E})dv = \int \operatorname{div}(V\epsilon\mathbf{E})dv - \int \epsilon\mathbf{E}\nabla V dv$$

and

$$\int \operatorname{div}(V\epsilon\mathbf{E})dv = -\sum\int V'\epsilon'\mathbf{E}' dS_i - \int (V'\epsilon'\mathbf{E}'\mathbf{n}' + V''\epsilon''\mathbf{E}''\mathbf{n}'')dS_0,$$

where S_0 denotes the surfaces of discontinuity in the dielectric, which in the integration by parts must be taken together with the surfaces of the conductors as the boundary of the realm of integration. Hence U can be written

(6) $$U = \int \epsilon(\tfrac{1}{2}\mathbf{E}^2 + \mathbf{E}\nabla V)dv + \sum\int (V' - V_i)\epsilon'\mathbf{E}'\mathbf{n}' dS_i + \int (V'\epsilon'\mathbf{E}'\mathbf{n}' + V''\epsilon''\mathbf{E}''\mathbf{n}'')dS_0$$

and the condition $\delta U = 0$ for all admissible variations of $\mathbf{E}$ gives the required equations.

By variations $\delta\mathbf{E}$ which vanish at the surfaces S_i and S_0 the first term only of U is affected and gives the condition $\mathbf{E} = -\nabla V$, so that the vector function $\mathbf{E}$ must have a scalar potential V, which according to (I) must satisfy the generalized Poisson equation

(7) $$\operatorname{div}(\epsilon\nabla V) + \rho = 0.$$

Then with this condition satisfied variations of **E** which vanish at S_0 but not at the surfaces S_i affect only the second term of U and give the condition $V = V_i$, making the potential constant over the surface of each single conductor ; and finally variations which at the surfaces S_0 are restricted only by the boundary condition (II) give the condition $V' = V''$, making the potential continuous even at surfaces where its derivatives may be discontinuous.

CHICAGO, ILL.,
May, 1908.

THE FOURTH INTERNATIONAL CONGRESS OF MATHEMATICIANS.

THE fourth international congress of mathematicians was held at Rome, April 6 to 11, 1908, under the efficient management of the Circolo Matematico di Palermo and under the patronage of His Majesty, the King of Italy. Including the ladies who accompanied the members of the congress, the enrollment was more than seven hundred. The list contains the names of the following Americans: Miss E. M. Coddington, H. W. Curjel, E. W. Davis, T. S. Fiske, A. B. Frizell, W. J. Graham, J. G. Hardy, E. A. Harrington, A. S. Hawkesworth, T. F. Holgate, A. Macfarlane, Artemas Martin, C. L. E. Moore, E. H. Moore, Simon Newcomb, G. D. Olds, G. B. Pegram, D. E. Smith, J. M. Van Vleck, W. D. A. Westfall.

The general order of the program provided for sectional meetings in the morning and general conferences in the afternoon. However this order was broken occasionally. The arrangements of the committee on entertainment left nothing to be desired.

The first meeting of the members of the congress was at the reception offered by Professor Tonelli, Rector of the University of Rome, Sunday evening, April 5. This was the beginning of the social part of the occasion. Those who attended the congress will always have pleasant recollections of these receptions and other entertainments. The opening reception was held in the library of the University. The Mayor of the city of Rome, Mr. Nathan, was present. Acquaintances which were to broaden during the week began here. Excellent refreshments were served, and the mathematicians showed that they were not incapable of enjoying this part of the program.