THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fortieth regular meeting of the Society was held in New York City on Saturday, October 31, 1908, extending through a single morning session. The attendance included the following twenty-one members:

Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor James Maclay, Mr. H. F. MacNeish, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. L. P. Siceloff, Professor D. E. Smith, Dr. Elijah Swift, Professor H. D. Thompson, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. K. Whittemore.

The President of the Society, President H. S. White, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor J. A. Brewster, St. Angela's College; Professor W. H. Butts, University of Michigan; Dr. C. F. Craig, Cornell University; Professor T. A. Martin, Mt. Union College; Professor M. T. Peed, Emory College; Mr. G. E. Roosevelt, New York, N. Y.; Mr. L. M. Saxton, College of the City of New York. Four applications for admission to membership in the Society were received.

A list of nominations of officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting.

It was decided to hold the annual meeting at Baltimore, on Wednesday and Thursday, December 30–31, in affiliation with the American association for the advancement of science.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the theory of functions of a triple variable."

(2) Professor R. D. CARMICHAEL: "Notes on the simplex theory of numbers."

(3) Professor EDWARD KASNER: "Conformality and functions of two or more complex variables (second paper)."

(4) Professor G. A. MILLER: "On the groups generated by two operators satisfying the equation $s_1s_2 = s_2^\alpha s_1^\beta$."

(5) Professor E. B. WILSON: "The number of types of collineations."

(6) Dr. FRANK IRWIN: "The invariants of linear differential expressions."

(7) Dr. A. E. LANDRY: "A geometrical application of binary syzygies."

Dr. Landry's paper was communicated to the Society through Professor White, Dr. Irwin's paper through Professor Veblen. In the absence of the authors the papers of Professor Carmichael, Professor Miller, Professor Wilson, and Dr. Landry were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael defines a system of triple numbers (an extension of ordinary complex numbers) and introduces functions of a triple variable thus defined. For these numbers all the formal laws of operation of ordinary complex algebra are found to hold. It follows then from a well-known theorem that the zero product theorem cannot hold in its general form; but it turns out that only a slight modification of this theorem is necessary and that, with the small exceptions to which this gives rise, algebraic operations with these triple numbers may be performed as with ordinary complex numbers.

In particular it is found that the Cauchy and Laurent expansions hold practically without modification. It becomes then a simple matter to write out the properties of holomorphic functions of this triple variable through their analogies with those in the function theory of ordinary complex numbers.

Numbers and operations in this triple system have a geometrical interpretation in relation to space as those in the complex system have in relation to the plane.

2. Professor Carmichael's second paper consists of three notes:

I. The first deals with the factors of the continued product of an arithmetical series of integers. A formula is found for the highest power of a prime p contained as a factor in this product. Some simple forms for special cases are worked out.

II. This note deals with an extension of Fermat's theorem, and a new function $M(a)$ of an integer a is defined. Its values for every possible $a \leqq 250$ are tabulated.

III. This states briefly a method for finding all the solutions of $\phi(z) = a$, where $\phi(z)$ denotes the totient of a.

3. In a paper presented at the summer meeting of the Society Professor Kasner discussed the group of transformations defined by pairs of analytic functions $\phi(z, w)$, $\psi(z, w)$ of two independent complex variables $z = x + iy$, $w = x' + iy'$, which were regarded as representing a point in four-dimensional space. In the present paper the variables z, w are taken to represent an ordered point pair or vector in the plane. The vector transformations which then arise are characterized by this property: If any three neighboring vectors are so situated that the infinitesimal triangles formed by their initial and terminal points are similar, the same is true of the transformed vectors (the new triangles are then necessarily similar to the original triangles). Extensions to n variables and to conjugate variables are easily made.

The results are applied to Poincaré's generalization of the Dirichlet problem (*Palermo Rendiconti*, 1907). As a supplement to Poincaré's discussion of three-dimensional varieties, it is shown that not all two-dimensional varieties are equivalent. Finally the linear transformations which appear in Picard's theory of hyperfuchsian functions are shown to admit of simple interpretation.

4. It is first pointed out by Professor Miller that all the groups of genus zero belong in the category discussed. He then takes up the relation $s_1s_2 = s_2^2s_1^2$, first studied by Cayley in 1878 and more recently by Netto in 1905, and greatly extends their results. Among the new theorems proved are the following: If two operators of order 6 satisfy the condition $s_1s_2 = s_2^2s_1^2$, the order of s_1s_2 cannot be odd unless one of these operators is the inverse of the other. When s_1s_2 is of order 2, 4, or 6, the groups generated by s_1, s_2 have orders which divide 24, 192, or 648 respectively, and they may actually have these orders. If two operators of orders 4 and 6 respectively satisfy the condition $s_1s_2 = s_2^3s_1^2$, they generate a group of order 48 involving the direct product of the alternating group of order 12 and the group of order 2. The quotient group of this group of order 48 with respect to the subgroup generated by the cube of the operator of order 6 is the symmetric group of order 24. If two operators of orders 4 and 12 respectively satisfy the condition

$s_1s_2 = s_2^3s_1^2$, they generate a group of order 96 involving the quaternion group as a characteristic subgroup.

The most important results obtained relate to the case where the two generating operators of a group satisfy the relation $s_1s_2 = s_2^{-2}s_1^{-2}$. It is first proved that either two such operators must have the same order or the order of one is three times that of the other. Moreover, the order of s_1s_2 must be odd, and the following relations are always true $(s_1s_2)^3 = s_1^3s_2^3$, $(s_1s_2)^{11} = s_1^5s_2^5$. By means of these relations a number of special cases can readily be considered. Some of these special results are expressed in the following theorem: If two operators of order 6 satisfy the condition $s_1s_2 = s_2^4s_1^4$, they generate one of the following groups: the cyclic group of order 6, the direct product of the group of order 2 and the non-cyclic group of order 9, the metacyclic group of order 42, or a group of order 126 involving an invariant cyclic subgroup of order 21.

5. In a short note, which has since been published in the *Jahresbericht der deutschen Mathematiker-Vereinigung*, on the number of types of collineations, Professor Wilson tabulates the number of types of non-singular collineations, classified according to the different systems of elementary divisors, when the number of homogeneous variables is twelve or less. It is found that the number of types is nearly doubled on passing from one dimension to the next higher throughout the range of dimensions here considered.

6. Dr. Irwin's paper deals with linear differential expressions of the nth order, both ordinary and partial, and their invariants under (1) change of dependent variable, (2) multiplication by a function of the independent variables, (3) change of independent variables. Cockle, in the *Philosophical Magazine*, 1870, handled the problem, for an ordinary differential equation under change of dependent variable, $u = \psi \cdot \eta$ by reducing the equation to a "canonical" form in which the coefficient of the $(n-1)$st derivative vanishes. If A be any coefficient of this canonical form, then A/ψ is a rational invariant, as well as $\psi^{-1}d^lA/d\psi^l$, and any invariant can be expressed as a function of these invariants. The writer finds that this method may be applied, with parallel results throughout, to ordinary differential expressions under change of independent variable. But, further, the attempt to apply the method to partial dif-

ferential expressions under change of dependent variable leads to results. Here we get for ψ the equations $\partial \log \psi / \partial x_i = K_i$, the K's being functions of the coefficients of the given differential expression, equations that are solvable only if the invariants $\partial K_i / \partial x_j - \partial K_j / \partial x_i$ vanish. Nevertheless, if we calculate the higher derivatives of ψ from $\partial \psi / \partial x_i = K_i \psi$ *in any manner whatever* and substitute in any coefficient A of the transformed differential expression, A / ψ will be an invariant. We also have invariants analogous to the invariants $\psi^{-1} d^l A / d\psi^l$ of ordinary differential expressions.

Lastly, the possibility of making a differential expression of the second order self-adjoint by multiplying it by a function of the independent variables is found to depend on the identical vanishing of an expression covariant for all three transformations.

7. Given a curve $J^{(m+1)}$ of order $m + 1$ and having an m-fold point, and a fundamental involution I_1^{2m-1} set up thereon, the problem dealt with in Dr. Landry's paper is how to determine other curves, of the same type and with the same multiple point, which intersect the given curve either at the base points of the I_1^{2m-1} or in sets of points given by covariants of the form which vanishes for the base points.

Let it be required to find the curve which intersects the $J^{(m+1)}$ at the vanishing points of a $C_{k,n}$. If we use homogeneous coordinates x_0, x_1, x_2 and place the multiple point at the vertex (1, 0, 0) of the reference triangle, the equation of the $J^{(m+1)}$ will be

$$x_0 C_{m,m}(x_1, x_2) = C_{p,m+1}(x_1, x_2),$$

where $C_{m,m}$ is the canonizant of the fundamental form $C_{1,2m-1}$.

The sought curve will have an equation of the form

$$x_0 C_{q,n-m-1} = C_{r,n-m};$$

and on eliminating x_0 we find that the intersections are given by

$$C_{p,m+1} C_{q,n-m-1} - C_{m,m} C_{r,n-m} = 0,$$

that is, we must have identically

$$C_{k,n} I_{p+q-k} \equiv C_{p,m+1} C_{q,n-m-1} - C_{m,m} C_{r,n-m};$$

so that the problem leads to a syzygy among the covariants of the fundamental form.

Next, the form of the invariant multiplier is investigated, and it is shown that the eliminant of $C_{m,m}$ and $C_{p,m+1}$ is a perfect mth power, of which I_{p+q-k} is the mth root. Lastly, the special cases of the quintic and septimic are discussed. The first case is treated exhaustively; for the second, on the other hand, only a summary of the main results obtained, without proofs, is given.

The paper will be published in the *Proceedings of the American Academy of Arts and Sciences.*

F. N. COLE,
Secretary.

ON THE GROUPS GENERATED BY TWO OPERATORS SATISFYING THE CONDITION $s_1s_2 = s_2^{-2}s_1^{-2}$.

BY PROFESSOR G. A. MILLER.

(Read before the Southwestern Section of the American Mathematical Society, November 28, 1908).

§ 1. *Introduction.*

REMARKABLE general properties may easily be proved as regards the system of groups generated by two operators s_1, s_2 which satisfy the condition expressed by one of the following pair of equivalent equations:

$$s_1s_2 = s_2^{-2}s_1^{-2}, \quad s_1^2s_2^2 = s_2^{-1}s_1^{-1}.$$

From the facts that $(s_2s_1)^2 = s_2s_1s_2s_1 = s_2^{-1}s_1^{-1} = (s_1s_2)^{-1}$ and that s_1s_2 is of the same order as s_2s_1, it results that the order of s_1s_2 is an odd number. From the same equations it follows that s_1s_2 is transformed into a power of itself both by s_1 and by s_2, and hence the cyclic group generated by s_1s_2 is invariant under the group G generated by s_1, s_2. As G is generated both by s_1s_2, s_1 and also by s_1s_2, s_2 it results from the preceding sentence that if s_1^{α} is the lowest power of s_1 which occurs in the cyclic group generated by s_1s_2 it is necessary that s_2^{α} is the lowest power of s_2 occurring in this group and vice versa. Moreover, if s_1 is commutative with $(s_1s_2)^n$ the following equations are satisfied:

$$(s_2s_1)^n = (s_1s_2)^n = (s_2s_1)^{-2n} \quad \text{since} \quad (s_2s_1)^2 = s_2^{-1}s_1^{-1} = (s_1s_2)^{-1}.$$