to Weierstrass, with the totality of those functions δy of class C' which vanish at x_1 and x_2 and satisfy the relation $\delta K = 0$.

The proof of this lemma — which is an essential step in the chain of conclusions, and whose omission forms a serious gap in the older theory — constitutes the second difficulty.

Neither of these difficulties occurs in the proof which we have given above.

FREIBURG, i. B.,
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NOTES ON THE SIMPLEX THEORY OF NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

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I. *Continued Product of the Terms of an Arithmetical Series.*

1. Let a and c be two relatively prime positive integers and form the arithmetical series

$$xa + c, \quad (x = 0, 1, 2, \cdots, n-1).$$

If we inquire what is the highest power of a prime p contained in the product

$$\prod_{x=0}^{x=n-1} (xa + c), \quad a \not\equiv 0 \pmod{p},$$

we shall find that the general result takes an interesting form. The solution of the problem may be effected in the following manner:

Evidently there exists some number x such that $xa + c$ is divisible by p. Let i be the smallest value of x for which this division is possible, and let c_1 be the quotient thus obtained. Using the notation

$$H\{y\} \tag{1}$$

to represent the index of the highest power of p contained in y, we will show that

$$H\left\{\prod_{x=0}^{x=n-1} (xa + c)\right\} = H\left\{\prod_{x=0}^{x=e_1} (xa + c_1)\right\} + e_1 + 1, \tag{2}$$

where

$$e_1 = \left[\frac{n-1-i_1}{p}\right]$$

is the largest integer not greater than $(n-1-i_1)/p$. In order to prove (2) we have only to notice that in the product of its first member only factors of the form

$$(mp+i_1)a+c$$

contain p and that the quotient of the division is always of the form

$$ma+c_1,$$

and that e_1 is the highest possible value of m. Performing the same operation on the H-function of the second member and continuing the process, we should finally arrive at a number which is simply the index of the required power of p.

In order to write this result in a convenient form let us define a suitable notation. Let i_r be the least integer such that $i_r a + c_{r-1}$ contains p and let c_r be the quotient of this division. For uniformity, set $c = c_0$ and $n-1 = e_0$. Further, let e_r be defined by

$$\left[\frac{e_{r-1}-i_r}{p}\right] = e_r. \tag{3}$$

Also let t be the first subscript for which

$$c_t(a+c_t)(2a+c_t)\cdots(e_t a+c_t)$$

does not contain the factor p. Then the preceding result may be written thus

$$H\left\{\prod_{x=0}^{x=n-1}(xa+c_0)\right\} = \sum_{r=1}^{r=t-1}(e_r+1). \tag{4}$$

Since $0 \leqq i_r \leqq p-1$, as is evident from the definition of i_r, we may deduce from (3) the following inequalities:

$$\left[\frac{e_{r-1}-(p-1)}{p}\right] \leqq e_r \leqq \left[\frac{e_{r-1}}{p}\right].$$

Hence

$$\left[\frac{e_{r-1}+1}{p}\right] \leqq e_r + 1 \leqq \left[\frac{e_{r-1}+p}{p}\right]. \tag{5}$$

This gives

$$\left[\frac{n}{p}\right] \leqq e_1 + 1 \leqq \left[\frac{n-1}{p}\right] + 1,$$

$$\left[\frac{n}{p^2}\right] \leqq e_2 + 1 \leqq \left[\frac{n-1}{p^2}\right] + 1,$$

$$\left[\frac{n}{p^3}\right] \leqq e_3 + 1 \leqq \left[\frac{n-1}{p^3}\right] + 1,$$

$$\cdots\cdots\cdots$$

Taking the sum of these inequalities, we have by (4)

$$(6)\quad \begin{aligned}\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots &\leqq H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\}\\ &\leqq \left[\frac{n-1}{p}\right] + \left[\frac{n-1}{p^2}\right] + \cdots + R(n-1),\end{aligned}$$

where $R(n-1)$ is the index of the highest power of p not greater than $n-1$.

This result takes different forms according as n is or is not a power of p. If n is a power of p, we have evidently

$$(7)\qquad \left[\frac{n}{p^a}\right] = \left[\frac{n-1}{p^a}\right] + 1$$

for every p^a equal to or less than n. Remembering that when $n = p^h$

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = \frac{p^h - 1}{p-1},$$

and using equation (7) in connection with inequality (6), we have

$$(8)\qquad H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\} = \frac{n-1}{p-1}, \quad n = p^h.$$

When n is not a power of p, it is evident that

$$(9)\qquad \left[\frac{n}{p^a}\right] = \left[\frac{n-1}{p^a}\right].$$

Suppose now that

$$(10)\qquad n = \delta_h p^h + \delta_{h-1}p^{h-1} + \cdots + \delta_1 p + \delta_0, \quad \delta_h \neq 0,$$

and at least one other δ is not zero. Employing (9) and the well-known formula

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(\delta_h+\delta_{h-1}+\cdots+\delta_1+\delta_0)}{p-1},$$

we may write (6) as follows:

$$(11)\qquad \begin{aligned}\frac{n-(\delta_h+\cdots+\delta_1+\delta_0)}{p-1}&\leqq H\left\{\prod_{x=0}^{x=n-1}(xa+c_0)\right\}\\ &\leqq h+\frac{n-(\delta_h+\cdots+\delta_1+\delta_0)}{p-1}.\end{aligned}$$

The inequalities in (11) confine the value of H in narrow limits which are easily calculated.

2. In the series $xa+c$, it may happen that the first x for which $xa+c$ is divisible by p will give c as the quotient of this division. Then in the preceding discussion all the c's are equal; and then also all the i's. Dropping subscripts from i and c and making repeated use of equation (3), we have

$$e_1=\left[\frac{n-1-i}{p}\right],$$

$$e_2=\left[\frac{e_1-i}{p}\right]=\left[\frac{e_1p-ip}{p^2}\right]=\left[\frac{n-1-i-ip}{p^2}\right],$$

$$e_3=\left[\frac{e_2-1}{p}\right]=\left[\frac{e_2p^2-ip^2}{p^3}\right]=\left[\frac{n-1-i-ip-ip^2}{p^3}\right],$$

. . . .

If we add one to each member of each of these equations and take the sum of the results; then further, if we replace the resulting first member by its value taken from (4), we have

$$(12)\qquad \begin{aligned}H\left\{\prod_{x=0}^{x=n-1}(xa+c)\right\}&=\left[\frac{n-1-i+p}{p}\right]\\ &+\left[\frac{n-1-i-ip+p^2}{p^2}\right]+\left[\frac{n-1-i-ip-ip^2+p^3}{p^3}\right]+\cdots.\end{aligned}$$

3. If $a=c=1$, equation (12) takes a very simple form. For this case $i=p-1$. The result is the well-known theorem that the highest power of p contained in $n!$ is that whose index is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(s_h+\cdots+s_1+s_0)}{p-1},$$

where

$$n = s_h p^h + s_{h-1} p^{h-1} + \cdots + s_1 p + s_0.$$

4. If $a = 2$ and $c = 1$, equation (12) takes a special form of considerable interest. The terms of $xa + c$ are the natural odd numbers in order, and p is an odd prime. It is evident that $i = \frac{1}{2}(p - 1)$. Therefore

$$\left[\frac{n - 1 - i - ip \cdots - ip^{\beta-1} + p^\beta}{p^\beta}\right]$$

$$= \left[\frac{2n-2-2i-2ip-\cdots-2ip^{\beta-1}+2p^\beta}{2p^\beta}\right] = \left[\frac{2n-1+p^\beta}{2p^\beta}\right].$$

Then (12) becomes

$$H\{1 \cdot 3 \cdot 5 \cdots (2n-1)\} = \left[\frac{2n-1+p}{2p}\right] + \left[\frac{2n-1+p^2}{2p^2}\right] + \left[\frac{2n-1+p^3}{p^3}\right] + \cdots.$$

II. *An Extension of Fermat's Theorem.*

It will be shown that the congruence

$$x^{\phi(n)} \equiv 1 (\text{mod } n),$$

where $\phi(n)$ is Euler's ϕ-function of n, is still true when the modulus is a multiple of n formed in a definite way, x being prime to the new modulus.

It has been shown * that $\phi(z) = a$ has always more than one solution. If z_1 and z_2 are two roots of $\phi(z) = a$, then z_1 and z_2 must each have a factor not common to the two except when one is an odd number and the other is twice that odd number; and hence, except in this case, their lowest common multiple is greater than either of them. Now if $z_1, z_2, \cdots, z_i$ are all the roots of $\phi(z) = a$, we have by Fermat's theorem the congruences

$$x^a \equiv 1 (\text{mod } z_1),\ x^a \equiv 1 (\text{mod } z_2),\ \cdots,\ x^a \equiv 1 (\text{mod } z_i),$$

where in each case x is prime to the modulus involved. Now if L is the lowest common multiple of $z_1, z_2, \cdots, z_i$ and x is prime to L, we have

(1) $$x^a \equiv 1 (\text{mod } L),$$

where L is greater than any number whose totient is a except

* Carmichael, BULLETIN, vol. 13, p. 241.

when the equation $\phi(z) = a$ has only the two solutions $z = L$, $z = \frac{1}{2}L$. Hence,

THEOREM. *Except when n and $\frac{1}{2}n$ are the only numbers whose totient is the same as that of n, the congruence $x^{\phi(n)} \equiv 1$ holds for a modulus which is some multiple of n.*

A working method for finding such a modulus is the following:

Set $\phi(n) = a$, for convenience. Separate a into its prime factors and find the highest power of each prime p contained in a such that $\phi(p^{\alpha})$ is equal to or is a factor of a. Suppose that the following primes are found: $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_j^{\alpha_j}$. Then write out all the divisors of a and take every prime q such that $q - 1$ is equal to any one of these divisors, but $q \neq$ any p; and say we have $q_1, q_2, \cdots, q_k$. Then set

(2) $$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j} q_1 q_2 \cdots q_k.$$

Then evidently

(3) $$X^a \equiv 1 (\text{mod } M),$$

when X is prime to M. (It should be noticed that M may be a multiple of L in congruence (1).)

As thus defined, M is a definite function of a; say $M = M(a)$. For every odd value of a, except $a = 1$, we have $M(a) = 1$, as the reader may readily verify. Some even values of a give also $M(a) = 1$. There follows a table giving the value of $M(a)$ for each a for which $M \neq 1$ up to $a = 150$.

a	$M(a)$	a	$M(a)$	a	$M(a)$
2	12	48	2 227 680	104	12 720
4	120	52	6 360	106	1 284
6	252	54	43 092	108	22 265 704 680
8	240	56	6 960	110	33 396
10	132	58	708	112	26 740 320
12	32 760	60	3 407 203 800	116	7 080
16	8 160	64	32 640	120	279 390 711 600
18	14 364	66	388 332	126	549 092 628
20	6 600	70	9 372	128	65 280
22	276	72	10 087 262 640	130	17 292
24	65 520	78	948	132	50 483 160
28	3 480	80	18 400 800	136	10 960
30	85 932	82	996	138	1 646 316
32	16 320	84	285 962 040	140	13 589 400
36	69 090 840	88	491 280	144	342 966 929 760
40	108 240	92	5 640	148	17 880
42	75 852	96	432 169 920	150	12 975 732
44	2 760	100	3 333 000		
46	564	102	25 956		

III. *The Solutions of* $\phi(z)=a$.

It is desirable to have a general method for finding all the solutions of

$$\phi(z)=a$$

for any given a. The method used in Note II for finding M in congruence (1) is suggestive, and we may formulate a rule thus:

Find M as in Note II. Evidently, *the solutions of* $\phi(z)=a$ *will all be factors of M. Then examine all the factors of M and retain each one whose totient is a.*

ALABAMA PRESBYTERIAN COLLEGE,
ANNISTON, ALABAMA.

THE SOLUTION OF BOUNDARY PROBLEMS OF LINEAR DIFFERENTIAL EQUATIONS OF ODD ORDER.

BY PROFESSOR W. D. A. WESTFALL.

E. SCHMIDT[1] has studied the set of linear integral equations with non-symmetric matrix

$$(1)\quad \phi_i(s)=\lambda_i\int_a^b K(s,t)\psi_i(t)dt,\qquad \psi_i(s)=\lambda_i\int_a^b K(t,s)\phi_i(t)dt,$$

and has shown that, if there can be found for a function $f(x)$ a continuous function $h(x)$, such that

$$(2)\qquad f(x)=\int_a^b K(x,t)\,h(t)dt,$$

then

$$(3)\qquad f(x)=\sum_i\frac{\phi_i(x)}{\lambda_i}\int_a^b h(t)\,\psi_i(t)dt,$$

where ϕ_i runs over a complete set of solutions of (1) which have been normalized and orthogonalized, *i. e.*,

$$(4)\qquad \int_a^b\phi_i\psi_j dx=\begin{Bmatrix}1, & i=j,\\ 0, & i\neq j.\end{Bmatrix}$$

* *Math. Annalen*, vol. 63, p. 459.