III. *The Solutions of* $\phi(z) = a$.

It is desirable to have a general method for finding all the solutions of

$$\phi(z) = a$$

for any given a. The method used in Note II for finding M in congruence (1) is suggestive, and we may formulate a rule thus:

Find M as in Note II. Evidently, *the solutions of $\phi(z) = a$ will all be factors of M. Then examine all the factors of M and retain each one whose totient is a.*

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THE SOLUTION OF BOUNDARY PROBLEMS OF LINEAR DIFFERENTIAL EQUATIONS OF ODD ORDER.

BY PROFESSOR W. D. A. WESTFALL.

E. SCHMIDT[1] has studied the set of linear integral equations with non-symmetric matrix

$$(1) \quad \phi_i(s) = \lambda_i \int_a^b K(s, t)\psi_i(t)dt, \qquad \psi_i(s) = \lambda_i \int_a^b K(t, s)\phi_i(t)dt,$$

and has shown that, if there can be found for a function $f(x)$ a continuous function $h(x)$, such that

$$(2) \qquad f(x) = \int_a^b K(x, t)\, h(t)dt,$$

then

$$(3) \qquad f(x) = \sum_i \frac{\phi_i(x)}{\lambda_i} \int_a^b h(t)\, \psi_i(t)dt,$$

where ϕ_i runs over a complete set of solutions of (1) which have been normalized and orthogonalized, *i. e.*,

$$(4) \qquad \int_a^b \phi_i \psi_j dx = \begin{Bmatrix} 1, & i = j, \\ 0, & i \neq j. \end{Bmatrix}$$

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* *Math. Annalen*, vol. 63, p. 459.

A connection can be established with the boundary problem of differential equations as follows:

If the linear differential expression of odd order $L(y)$ is equal to its adjoint with negative sign, it satisfies the relation

$$(5) \quad \int_a^b [zL(y) + yL(z)]\,dx = [P(y, z)]_a^b + [p_0(zy^{(n-1)} + yz^{(n-1)})]_a^b$$

$$(6) \qquad L(y) = p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y',$$

provided the p, y, and z satisfy certain continuity restrictions. $P(y, z)$ contains no derivatives of y or z of order higher than the $(n-2)$nd.

It is known * that the Green's function of $L(y)$ is skew-symmetric,

$$(7) \qquad G(s, t) + G(t, s) = 0,$$

and that no characteristic solution of $L(y) + \lambda_i y = 0$ exists for a real value of λ. Here $p_0, p_1, \cdots, p_{n-1}$ are real functions of the real variable x.

Let $\lambda_k = l_k + in_k$ be a complex value of λ for which there exists the characteristic solution

$$u_k(x) = \phi_k(x) + i\psi_k(x)$$

of $L(y) + \lambda_k y = 0$. Then $\bar{u}_k = \phi_k(x) - i\psi_k(x)$ is a characteristic solution of $L(y) + \bar{\lambda}_k y = 0$, $\bar{\lambda}_k = l_k - in_k$. Substitute $u = u_k$, $v = \bar{u}_k$ in (5).

$$2l_k \int_a^b u_k(x)\bar{u}_k(x)dx = 0.$$

Hence $l_k = 0$. Since u_k is a solution of $L(y) + in_k y = 0$, we have

$$L(\phi_k + i\psi_k) + in_k(\phi_k + i\psi_k) = 0,$$

or

$$(8) \qquad L(\phi_k) - n_k\psi_k = 0, \qquad L(\psi_k) + n_k\phi_k = 0.$$

Since u_k is a characteristic solution, ϕ_k and ψ_k constitute a set of characteristic solutions of equations (8). The relation (5) shows us that these satisfy with the Green's function $G(x, t)$ the equations

$$\phi_k(x) = n_k \int_a^b G(t, x)\psi_k(t)dt, \qquad \psi_k(x) = -n_k \int_a^b G(t, x)\phi_k(t)dt.$$

* The author's dissertation, Göttingen, 1905.

Or from (7)

$$(9)\quad \phi_k(x) = n_k \int_a^b G(t,x)\,\psi_k(t)dt, \qquad \psi_k(x) = n_k \int_a^b G(x,t)\,\phi_k(t)dt.$$

Moreover the solutions of equations (9), considered as integral equations with the known matrix $G(x, t)$, give a set of characteristic solutions of (8). This establishes the relation between Schmidt's pair of integral equations and the linear differential equation.

If $f(x)$ is continuous with its first n derivatives and satisfies the boundary conditions satisfied by the Green's function, equation (2) is solved by differentiation

$$L(f) = -h(x).$$

Hence there are an infinite number of pairs of solutions of (8), and an infinite number of characteristic solutions of $L(y) + in_k y = 0$.

COLUMBIA, MO.,
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A CLASS OF FUNCTIONS HAVING A PECULIAR DISCONTINUITY.

BY PROFESSOR W. D. A. WESTFALL.

CONSIDER all functions discontinuous for all rational values of the independent variable, and continuous and equal to zero for all irrational values. They are of the form

$$f\left(\frac{p}{q}\right) \neq 0, \quad p \text{ and } q \text{ prime to each other,}$$

$$(1)\qquad f(\alpha) = 0, \quad \text{for } \alpha \text{ irrational, with the condition that}$$

$$\lim_{q=\infty} f\left(\frac{p}{q}\right) = 0.$$