Or from (7)

$$(9)\quad \phi_k(x) = n_k \int_a^b G(t, x)\,\psi_k(t)dt, \qquad \psi_k(x) = n_k \int_a^b G(x, t)\,\phi_k(t)dt.$$

Moreover the solutions of equations (9), considered as integral equations with the known matrix $G(x, t)$, give a set of characteristic solutions of (8). This establishes the relation between Schmidt's pair of integral equations and the linear differential equation.

If $f(x)$ is continuous with its first n derivatives and satisfies the boundary conditions satisfied by the Green's function, equation (2) is solved by differentiation

$$L(f) = -h(x).$$

Hence there are an infinite number of pairs of solutions of (8), and an infinite number of characteristic solutions of $L(y) + in_k y = 0$.

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A CLASS OF FUNCTIONS HAVING A PECULIAR DISCONTINUITY.

BY PROFESSOR W. D. A. WESTFALL.

CONSIDER all functions discontinuous for all rational values of the independent variable, and continuous and equal to zero for all irrational values. They are of the form

$$f\left(\frac{p}{q}\right) \neq 0, \quad p \text{ and } q \text{ prime to each other,}$$

$$(1)\qquad f(\alpha) = 0, \quad \text{for } \alpha \text{ irrational, with the condition that}$$

$$\lim_{q=\infty} f\left(\frac{p}{q}\right) = 0.$$

The following are examples of such functions:

$$(2)\qquad \phi_n\left(\frac{p}{q}\right)=\frac{1}{q^n},\quad \phi_n(1)=1,\quad \phi_n(\alpha)=0,$$

$$(3)^*\qquad \phi\left(\frac{p}{q}\right)=\frac{1}{q\,!},\quad \phi(1)=1,\quad \phi(\alpha)=0,$$

$$(4)\qquad \psi\left(\frac{p}{q}\right)=\frac{1}{q^q},\quad \psi(1)=1,\quad \psi(\alpha)=0.$$

Liouville has shown † that if α is an algebraic irrationality of the nth order

$$\left|\frac{p}{q}-\alpha\right|>\frac{A}{q^n},$$

where A is independent of q. Hence ϕ_n has a zero derivative at all algebraic irrationalities of order less than n, and ϕ and ψ at all algebraic irrationalities.

It will now be shown that it is impossible for a function of type (1) to have a derivative for every irrational value of the independent variable. Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of rational numbers such that

$$|a_n-a_{n-1}|<\frac{1}{|f(a_n)|}\quad\text{and}\quad<\left|\frac{1}{q^q}\right|,$$

if $a_n=p/q$. Then this defines a transcendental number α at which the difference quotient of $f(x)$ taken over the sequence a_n is greater than $\frac{1}{2}$. Hence a derivative cannot exist, since it is evidently zero if it exists. The definition of this point shows that such a point exists in every interval.

The above holds equally well for a function discontinuous at all rational points, and continuous at all irrational points in such a way that it coincides in these points with a function having a derivative throughout.

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* An example first used by Professor Osgood in his lectures.

† *Comptes Rendus* for 1844.