SYLVESTER'S MATHEMATICAL PAPERS.

The Collected Mathematical Papers of James Joseph Sylvester. (Edited by H. F. BAKER.) Volume I, xii + 650 pp. Volume II, xvi + 731 pp. Cambridge, University Press, 1904, 1908.

PRIOR to 1877 there had been made in this country a few mathematical investigations of permanent value, in particular that by Benjamin Peirce on linear associative algebras. But such investigations had been done in isolation, and independently, in a measure, of contemporary European activity. These early writers were prophets, as it were, preparing the way for an era of wide-spread activity and interest in mathematical research. Chief among the forces which inaugurated this era was the advent of Sylvester at the opening of John Hopkins University in 1877, and his founding of the *American Journal of Mathematics* the following year. His high ideals for the search after truth, and his unbounded enthusiasm for mathematical science not only proved a powerful stimulus for his colleagues and students, but reached out beyond his immediate surroundings to the country at large.

Sylvester's work in this period, however, does not belong properly to the present review, since the first two volumes of his papers relate to the years 1837–1873. There occur 178 titles, including a few reports or abstracts of communications to societies. These latter titles appear to have been omitted from the Royal Society index, which shows 150 titles down to the year 1873. To judge from the later titles, including eighty-one for the next decade, there will probably be two additional volumes.

Sylvester's investigations related chiefly to algebra, including in that term the theory of invariants, matrices, theory of equations, etc., and the related domains of number theory, differential invariants, etc. However, there are various papers on astronomy, mechanics and physics, while a later volume will include his paper, "An application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics." His earliest paper related to Fresnel's optical theory of crystals; the next two and a paper of 1850 relate to motion and rest of fluids and rigid bodies. In

1856, there is "A trifle on projectiles" and a letter on Professor Galbraith's construction for the range of projectiles. In 1860 he wrote on the pressure of "mathematical" earth, the "essential quality which differentiates it from actual vulgar earth being continuity."

In the *Philosophical Magazine* for 1866, he has a twenty-four page article entitled "Astronomical prolusions; commencing with an instantaneous proof of Lambert's and Euler's theorems, and modulating through a construction of the orbit of a heavenly body from two heliocentric distances, the subtended chord, and the periodic time, and the focal theory of cartesian ovals, into a discussion of motion in a circle and its relation to planetary motion." After quoting from Lagrange: "His (Lambert's) theorem merits the especial notice of mathematicians, both on its own account, and because it appears difficult to arrive at by algebraical processes; so that it may be ranked among the small number of those in which geometrical seems to have the advantage over algebraical analysis," Sylvester makes some comments which might be called heresy on the part of the future Savilian professor of geometry at Oxford (in succession to the famous number-theorist Henry Smith), viz., "In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead over analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite."

In 1866 there is a sequel to the Astronomical prolusions with an equally ambitious title, and later on a supplement to the sequel. In the same year there is a rather extensive paper on the motion of a rigid body acted on by no external forces, and a shorter paper with a similar title. Soon afterwards, Sylvester became much interested in linkages and the general question of the conversion of motion. His lecture before the Royal Institution on January 23, 1874, "On recent discoveries in mechanical conversion of motion," brought to general notice the various possibilities of Peaucellier's linkage and inspired the well-known work of Hart and Kempe.

Sylvester's papers on geometry are not numerous. In 1850 there are elementary papers on the intersections and contacts of conics, and an "instantaneous demonstration of Pascal's theorem." In 1851 there is an important paper on the "contacts of lines and surfaces of the second order," in which he discusses the reduction of a pair of quadratic forms U, V in two, three or four variables, notes explicitly the invariance of the highest common factor of the minors of order r of the determinant of $U + \lambda V$, and makes use of the multiplicity of the linear factors in classifying the forms. This work forecasts in a manner the more complete theory of Weierstrass of 1868. In 1854 there is a supplementary note giving certain corrections. In volume I, pages 378–381, is given Sylvester's paper of 1852, in which he enunciates his "Law of inertia for quadratic forms," which states the invariance of the number of positive coefficients in the form $\sum A_i \zeta_i^2$ to which a real quadratic form may be reduced by a real linear transformation. His Probationary lecture on geometry, delivered in 1854, is given on the opening pages of volume II. On pages 236–244 are reproduced Sylvester's notes in the *Comptes Rendus* for 1861 on the involution of six lines in space, "double-sixes" of lines, and the construction of the 27 lines on a cubic surface. Two years later he considered questions of centers of gravity and the "principles of barycentric perspective."

Five notes or papers on the successive involutes to a circle appeared in 1868 and 1869. Adopting Cayley's name cyclode for the nth involute of a circle, Sylvester treated at length the reducible cyclodes in his paper in the *Proceedings of the London Mathematical Society* for 1869. He pursued this subject with the greatest enthusiasm. He remarks that "As crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich spiral has sprung the vast and interminable calculus of cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of denumeration, and by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the scriptural comparison of the kingdom of heaven to a grain of mustard-seed," etc. He concludes his article with a formidable list of prerequisites to the mastering of his exposition of the theory.

The theory of numbers proved attractive to Sylvester throughout his sixty years of scientific activity. His first notes related to Wilson's theorem. In 1847 he discussed certain ternary cubic equations (recurring to the subject in 1879 and 1880). In the first of this series of papers, he took occasion to correct some errors in an earlier, hastily written, communication to the Institute of France. In the second of the series, he states that certain of his friends on the continent have "complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the theory of elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication." The third of the series is entitled "On the *general* solution (in certain cases) of the equation $x^3 + y^3 + Az^3 = Mxyz$." Sylvester begins as follows, "I shall restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify." In 1854 there was a further note on Wilson's theorem. In 1857 there were two papers on the partition of numbers, another on the resolvability of any integer into the sum of four squares, and a paper developing an idea of Eisenstein. The next year he wrote "On the Problem of the Virgins, and the general theory of compound partition," declaring himself at a loss to conjecture why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred by Euler to "the rule of the virgins," unless it has some mystical reference to the alligation of the coefficients of the two equations; he quotes from some correspondence with De Morgan on this point. In 1859, Sylvester delivered at King's College seven lectures on the partitions of numbers. Outlines of these lectures were circulated privately at the time, but were first published in 1897 without revision by the author in the *Proceedings of the London Mathematical Society*. These outlines appear in the present volume II, pages 119–175. In the example on 120 there seems to be no reason for the omission of the partition 3, 1, 1. Cayley and Sylvester appear to have become interested in the subject on account of its application to invariants; the subject had previously engaged the attention of Euler, Waring, De Morgan, Herschel, and others. Since Sylvester's elaborate memoir on partitions in volume 5 of the *American Journal of Mathematics* belongs to a later period than the papers under review, the entire subject may appropriately be left to the care of the reviewer of the subsequent volumes. In 1860 and 1861 there were ten short papers on topics in the theory of numbers. In 1871 he discussed the Goldbach-Euler problem of the partition of an even number into two primes (recurring to the subject in later years), also some special cases of Dirichlet's theorem on the primes in an arithmetical progression.

The theory of substitution groups engaged Sylvester's attention in 1861. There were two notes in the *Comptes Rendus* relating to the number of substitutions of a certain kind. The note in the *Philosophical Magazine* related to an unsymmetrical six-valued function of six letters whose discovery had been ascribed by Cauchy to Hermite; Sylvester lays claim to priority in view of his paper of 1844 in the same journal, "On the principles of combinatorial aggregation."

Questions on tactic, a subject pursued nowadays in view of the applications in the algebraic theory of equations and in the foundations of geometry, were considered by Sylvester in the two papers last cited and three further papers of 1861 in the same journal. The results bear on substitution groups on nine letters and indirectly on the fifteen schoolgirl problem.

To the theory of determinants Sylvester made some well-known contributions which appeared in the *Philosophical Magazine* for 1851, and in the *Cambridge and Dublin Journal* for 1853. H. F. Baker, the editor of the present volumes, gives another exposition in present-day notations of Sylvester's main theorems on determinants (note to volume I).

Linear equations in finite differences were considered by Sylvester in 1862 in the *Comptes Rendus* and the *Philosophical Magazine.* In 1869 he gave in the latter journal "The story of an equation in differences of the second order."

Sylvester's dialytic method of elimination has passed into the very elements of mathematical instruction. It appeared in the *Philosophical Magazine,* 1840, pages 132, 379; 1842, page 534; 1851, page 221.

Sturm's functions were among the first objects of Sylvester's interest, and continued long to engage his attention. In 1841 he proceeded "to lay bare the internal anatomy of these remarkable forms." Divesting them of square factors, he obtains expressions in terms of the roots,

$$f_2 = \sum(k - l)^2(x - a)(x - b) \cdots (x - h), \text{ etc.},$$

which Cayley characterized as "singularly elegant." In resuming the subject in 1853, he built up a more general theory which relates, not merely to $f(x)$ and $f'(x)$, but to any pair of polynomials $f(x)$ and $\phi(x)$. To these he applies Sturm's process of the greatest common measure and obtains functions which serve to indicate the manner in which the roots of $f(x) = 0$ are intercalated among the roots of $\phi(x) = 0$. This paper on the "syzygetic relations," etc., is perhaps Sylvester's most elaborate contribution; it concludes with a seven page "Glossary of new or unusual terms, or of terms used in a new or unusual sense, in the preceding memoir." Many of these terms are familiar to present students of invariant theory. Sylvester has said: "Perhaps I may without immodesty lay claim to the appellation of the mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all other mathematicians of the age combined." It has been remarked by Forsyth that Sylvester drew almost entirely upon Latin for new names, while Cayley as invariably drew upon Greek. To the same year, 1853, belong the two papers "On a remarkable modification of Sturm's theorem," *Philosophical Magazine*, where a certain series of quotients, arising from the continuedfraction development of f'/f is shown to be signaletically equivalent with Sturm's series of residues; also a paper in which these quotients are expressed in terms of the linear factors of $f(x)$. "Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much wished for representation, with which I propose now and for ever to take my farewell of this long and deeply excogitated theorem."

Newton's rule for the discovery of imaginary roots and an inferior limit to their number had remained undemonstrated, notwithstanding the attempts by Maclaurin, Campbell, Waring, Euler, Lagrange, and others. In 1864, Sylvester published a memoir of about a hundred pages in which he established the rule for equations of degree five or less, and gave "a complete invariantive determination of the character of the roots of the general equation of the fifth degree." The next year he announced a proof for the degrees six and seven of this rule "so long the wonder and opprobrium of algebraists." In June of the same year 1865, Sylvester was rewarded by the discovery of a general proof and obtained a generalization of the rule. He contests with characteristic vigor the priority claim of J. R. Young. In 1866, Sylvester made a slight improvement in the statement of the generalized rule. The next year he presented "Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile work, and the theory of numbers." MacMahon states that one anallagmatic design by Sylvester was put down in the hall of the Junior United Service Club in Charles Street, Haymarket, but was unfortunately removed later whilst the hall was undergoing repair.

Reference should also be made to the elementary proof of Newton's rule which Sylvester gave in 1871. The theory of multiple roots had been investigated by Sylvester in 1841 and 1852; his theory for the limits to the real roots, given in 1853, connected with his investigation of Sturm's functions.

Canonical expressions for binary forms of even and odd degrees were first given by Sylvester in 1851 in three papers, of which the second was printed privately. To him the canonical form is "the very palpitating heart of the function laid bare to inspection." Cayley had previously reduced the general quartic to $x^4 + 6mx^2y^2 + y^4$. Employing the latter form, Sylvester proved in 1853 that every invariant of the general binary quartic is a rational integral function of the two well-known invariants of degrees 2 and 3, and similarly that Aronhold's S and T form a fundamental system ("scale," according to Sylvester) of invariants of the ternary cubic form. The subject of canonical forms has since been extended by Gundelfinger, Hilbert, and others.

The name invariant was introduced by Sylvester in 1851, *Philosophical Magazine*, II, page 396. Cayley's term had been hyperdeterminant.

Following the discovery of special invariants and covariants by Boole and Eisenstein, Cayley undertook a systematic study of invariants in 1845, and discovered processes for building invariants. In 1850 Hermite introduced his formes adjointes. These were obtained independently and from a simpler standpoint by Sylvester in 1851 in his brief paper, "On the general theory of associated algebraical forms"; he employs the term contravariants. In the series of papers in the *Cambridge and Dublin Mathematical Journal*, 1852–1854, entitled "Principles of the calculus of forms," Sylvester carried the theory of invariants far beyond its previous stage, introducing concepts and and processes of first importance. Special mention must be made of his general processes for the construction of contravariants and concomitants, later designated as Überschiebungen by Gordan and made by him the foundation of the theory. Sylvester introduced and developed the class of concomitants of a system of forms known as combinants. In connection with Cayley, he developed the theory of commutants. Of the very few misprints noted, we may cite that for $1 \cdot 2 \cdot \cdots \cdot n$ on pages 305 and 306 of volume I.

Sylvester's extensive papers on reciprocants belong to a period subsequent to that covered by the present two volumes.

The tentative classification of the papers under review according to the topics applied mathematics, geometry, theory of numbers, substitution groups, tactic, determinants, finite differences, elimination, Sturm's functions, Newton's rule for imaginary roots, canonical forms and the theory of invariants, was employed by the reviewer to secure continuity of thought, even at the expense of historical sequence, and with the hope of calling the attention of specialists to Sylvester's papers in their fields. For the latter reason, mention has been made of many of his minor papers. Sylvester's most important contributions related to the theory of equations and invariants. Since his work in these subjects is so well-known, it was deemed unnecessary to go into details as fully as would be warranted, but quite sufficient to enumerate landmarks which would recall Sylvester's greatest achievements.

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HILTON'S FINITE GROUPS.

An Introduction to the Theory of Groups of Finite Order. By HAROLD HILTON, M.A. Oxford, Clarendon Press, 1908. xii + 236 pp.

THE theory of groups, which was first developed because of its relation to the solution of algebraic equations, today enters to a greater or less extent into the structure of many other de-