contravariants. In the series of papers in the *Cambridge and Dublin Mathematical Journal*, 1852–1854, entitled "Principles of the calculus of forms," Sylvester carried the theory of invariants far beyond its previous stage, introducing concepts and and processes of first importance. Special mention must be made of his general processes for the construction of contravariants and concomitants, later designated as Überschiebungen by Gordan and made by him the foundation of the theory. Sylvester introduced and developed the class of concomitants of a system of forms known as combinants. In connection with Cayley, he developed the theory of commutants. Of the very few misprints noted, we may cite that for $1 \cdot 2 \cdot \cdots \cdot n$ on pages 305 and 306 of volume I.

Sylvester's extensive papers on reciprocants belong to a period subsequent to that covered by the present two volumes.

The tentative classification of the papers under review according to the topics applied mathematics, geometry, theory of numbers, substitution groups, tactic, determinants, finite differences, elimination, Sturm's functions, Newton's rule for imaginary roots, canonical forms and the theory of invariants, was employed by the reviewer to secure continuity of thought, even at the expense of historical sequence, and with the hope of calling the attention of specialists to Sylvester's papers in their fields. For the latter reason, mention has been made of many of his minor papers. Sylvester's most important contributions related to the theory of equations and invariants. Since his work in these subjects is so well-known, it was deemed unnecessary to go into details as fully as would be warranted, but quite sufficient to enumerate landmarks which would recall Sylvester's greatest achievements.

L. E. DICKSON.

HILTON'S FINITE GROUPS.

An Introduction to the Theory of Groups of Finite Order. By HAROLD HILTON, M.A. Oxford, Clarendon Press, 1908. xii + 236 pp.

THE theory of groups, which was first developed because of its relation to the solution of algebraic equations, today enters to a greater or less extent into the structure of many other departments of mathematical thought, notably the theories of numbers, automorphic functions, algebraic substitutions, geometric transformations, and differential equations. Accordingly there is need of textbooks which will present the theory of abstract groups, not merely from the standpoint of permutation groups, but with a view to its many-sided applications to other concrete groups as well.

The book under review is evidently an attempt — and a very successful one it is — to meet this need, so far as groups of finite order are concerned. Heretofore the student has been obliged to depend, for the most part, either on books in which only abstract groups and permutation groups were considered, or on those in which groups were a secondary consideration in comparison with their applications and were therefore inadequately treated. In accordance with the well-known pedagogical principle that the concrete should always precede the abstract, it would seem that the theory of abstract groups should be copiously illustrated by examples chosen from the various kinds of concrete groups.

Moreover, for the beginner there is available plenty of illustrative material that is extremely elementary and easily comprehended. For instance, the additive and multiplicative groups of ordinary numbers, including the case where they are roots of unity; groups of translations; groups of rotations ot solids like the cylinder and the regular prism ; groups generated by reflections in the coordinate planes of a rectangular set ; the complement and supplement group in trigonometry ; the cross-ratio group of order 6 ; and the groups of classes of congruent integers with respect to a prime modulus, under addition and multiplication, respectively. All of these and many more like them are employed to advantage in the present text. Six of the earlier chapters are devoted entirely to concrete groups of three general kinds, permutation groups, substitution groups (chiefly groups of linear substitutions), and groups of movements. Throughout the remainder of the volume the contents of these chapters are constantly drawn upon for examples with which to illustrate and verify the general theorems.

In the development of the abstract theory a very careful and logical sequence of theorems is employed, long and intricate proofs are avoided, progress is made by short steps, and no step is taken before it is needed. As an instance in point the group concept itself is not introduced until the ground has been cleared for it in the first four chapters. At the end of each step the author takes a fresh look around at the mathematical landscape opening before him; thus very little of the scenery escapes him, so far as he travels. The only omission of interesting elementary material that occurs to me is that of the determination of the distinct abstract groups of low order, as given, for instance, by Levavasseur in his Enumérations.

The reader will be struck by the condensed style and compact notation used in the book and by the unusual arrangement of its contents, at least two-thirds of which, I should judge, is in the form of examples; and forty-three pages at the end are devoted to "hints for the solution of the examples." Some of these examples are simple illustrations of the foregoing theory or easy steps in advance, but others are more difficult and would seem to require, from the beginner's standpoint, more extensive hints for their solution than those given. Indeed, to judge by some of the examples, one might almost suppose that the book was intended to be neither a textbook for beginners nor a treatise, but rather an encyclopedic compendium of theorems without proofs. But this is by no means the impression produced by the volume as a whole. Many of the examples are referred to in the proofs of later theorems and are, therefore, necessary links in the chain of reasoning. In short, the book is an extreme example of the use of the Socratic method in the writing of textbooks, but nevertheless, in the hands of an energetic student it will certainly prove valuable and stimulating.

The reader will note that the chief properties of the elements which go to make up groups are studied in detail before their combination into groups is introduced. The term element, as defined on page 1 is, therefore, restricted to those mathematical entities which are capable of combining into groups. For instance, not all singular matrices (those whose determinants vanish) are elements in this sense.

A few elementary theorems are given on infinite groups and semigroups. In regard to the latter it does not seem to have been noticed that if A is an element of infinite order, then according to any of the definitions given by De Séguier, Dickson, or Hilton, respectively, a semigroup can be formed by selecting powers of A in very unusual and peculiar ways, for instance, by selecting the 10th, 15th, 20th, 21st, and all higher powers of A. In order to avoid such oddities it would only be necessary to make an additional postulate to the effect that if ν is the greatest common divisor of λ and μ then a semigroup which contains A^λ and A^μ must also contain A^ν.

The very latest discoveries, so far as they relate to the more fundamental parts of the subject, are embodied in the book. It is significant that commutators, which were first applied to finite groups by Miller in 1896, are introduced on page 4 and freely used throughout the book. As an evidence of the increasing vogue of the Galois field, the reader will observe that it is defined in the third chapter and that its properties are constantly used for illustrative purposes in the later chapters; also that such an elementary thing as the field of classes of congruent integers with respect to a prime modulus is defined merely as a special case of a Galois field. In examples 15–18, page 30, and example 3, page 32, certain theorems are stated concerning equations and polynomials whose coefficients are evidently intended to be integral marks of a Galois field, whereas they are incorrectly stated to be any marks of the field.

There is another oversight in example 11, page 70; it is true, as stated there, that an infinite abelian group is the direct product of a subgroup H formed by the elements of finite order and a subgroup K whose elements (besides 1) are of infinite order, but K does not, as stated there, contain *all* the elements of infinite order. The same slip was made by De Séguier.

It is to be noted that the definition of independent elements given on page 55 is not the usual one. According to this definition two elements of order 4 in the quaternion group, which are not inverse to each other, are independent, whereas they are usually regarded as mutually dependent, because they have the same square.

The condensation of the author's style sometimes tends to produce obscurity. On page 90, in examples 7 and 8, the reader is evidently supposed to understand that transitive groups are referred to, because the general subject of the section is k-ply transitive groups; but the insertion of the adjective transitive would seem to be required in order to avoid ambiguity.

Throughout the literature of the theory of groups there is a very general lack of precision in the use of the term subgroup; sometimes it includes identity and the entire group and sometimes not. Thus apparent contradictions are apt to arise. A striking instance is to be found on page 139 of Hilton's book, where example 3 contains the theorem that the central and commutant of any group are characteristic (subgroups), while the example just before it states that a simple group has no characteristic subgroups!

The group of automorphisms (or isomorphisms) of a given group is advantageously defined on pages 136–137 in a purely abstract manner and not by means of its representation as a permutation group, as is usually done. In the last chapter a very brief introduction to the theory of group characteristics is given, and the subject is approached from the proper direction, namely, that of the representation of an abstract group as an irreducible linear substitution group. This route leads directly to the heart of the matter and makes it easily accessible, as Schur has shown in his "Neue Begründung."

In nomenclature there are a number of improvements upon existing usage. Thus "prime power group" replaces "group whose order is a power of a prime"; "commutant," "central," "dicyclic," and "normaliser" are apparently borrowed from De Séguier; "permutation" replaces the commoner word "substitution," in order to avoid confusion with the more general operation denoted by the term substitution; "point group" is used to denote a group of geometric movements that leave a given point fixed, and not, as in some of the older books, an "equivalent system of points" (and therefore not a group at all) which are carried into one another by the movements of a group. Since the word "class" has been much overworked, it would seem that the author might profitably have followed his own suggestion and used "speciality" instead. In the use of the expression "normal subgroup" Weber's example is followed. The useful distinction recently made between isomorphism and automorphism (sometimes called holomorphism) is preserved; but it is not evident why an un-English coinage like "not-square" should be retained, when the well-recognized prefix "non" is so immediately available.

The following misprints were found: page 44, § 12, line 5, "the single point P' corresponds to P" for "P' corresponds to the single point P"; page 53, example 17, (ii), "some constant" for "constants"; page 64, last line, "g" for "h"; page 140, line 2, "$2n^2$" for "$2n^4$"; page 189, § 2, example 8, (i), "x" and "n" are interchanged; page 212, § 9, example 4, (ii), "$\Delta = 0$" for "$\Delta \neq 0$."

There is a good index and an appendix containing a list of problems still unsolved. The book can be heartily commended for three principal reasons, its embodiment of the results of the latest research, its carefully arranged sequence of theorems, and the emphasis which it lays on applications and illustrations.

ARTHUR RANUM.

SHORTER NOTICES.

Histoire des Mathématiques. Par W. W. ROUSE BALL. Edidition française revue et augmentée. Traduite sur la troisième édition anglaise par L. FREUND. Tome deuxième. Avec des additions de R. de Montessus. Paris, A. Hermann, 1907. vi + 271 pp. 8 francs.

IT was natural to hope that Lieutenant Freund would succeed better with his second volume of Ball than he did with the first,* since he received warnings enough at the hands of numerous reviewers. In a way the hope has been realized, although in the mere matter of translation and of accuracy there is, unfortunately, no improvement.

The valuable part of this second volume, as it appears in its French form, lies in the additions made hy M. R. de Montessus, 'docteur ès-sciences mathématiques, lauréat de l'Institut.' These are numerous, particularly with respect to French mathematics, and they materially increase the helpfulness of the work as a book of reference. The period treated, being that beginning with Newton and running to the close of the nineteenth century, is one of great interest from the standpoint of the Paris mathematicians, and it was natural that a Cambridge writer could hardly do full justice to the labors of writers like l'Hospital, Varignon, De Montmort, D'Alembert, Laplace, and several others whose works Mr. Ball has briefly described. Other continental authors that are hardly mentioned at all in the English edition have fairly adequate biographical notices in the translation, and on this account the additions of Dr. de Montessus are particularly helpful. Among the added names are Meusnier, Lhuillier, Agnesi, Lacroix, Malfatti, Delambre, Montucla, Laurent, Cournot, Genocchi, Betti, Puiseux, Bouquet, Codazzi, Faà di Bruno, Catalan, Brioschi, Casorati, Halphen, Wronski, Bertrand, Laguerre, Stieltjes, Clebsch, Dupin, Chasles, Bellavitis, Cremona, Beltrami, Poinsot, Tisserand, and several others, none of whom are treated, or who at most are merely men-

* See review in the BULLETIN, vol. 12, p. 309.