Diophantische Approximationen. Eine Einführung in die Zahlentheorie. Von HERMANN MINKOWSKI. Leipzig, Teubner, 1907. viii + 235 pp.

THE title of this original and highly interesting work is applied by the author to extensive classes of inequalities to be satisfied by integral values of the unknowns, the constants being any quantities; just as hitherto equations of the corresponding type have been designated with the name of Diophantus. In the first chapter, the author considers the rational approximation to one or more arbitrary real numbers, gives a new proof of the existence of integral solutions of $sx - ry = 1$, s and r being relatively prime integers, and establishes the theorem that there exist integral values, not all zero, of three variables such that each of the three given linear forms of determinant unity has a value not exceeding unity numerically. In the proof, due to a suggestion by Hilbert, the author introduces the concept of the minimum of a given system of linear forms for integral values of the variables and shows that the minimum changes continuously under continuous variation of the coefficients of the forms. An earlier proof by Minkowski, based upon geometric considerations, was given in his Geometrie der Zahlen, Leipzig, 1896.

Chapters 2 and 3, forming about half the book, relate to lattices; a Zahlengitter in two or three dimensions is defined to be the totality of sets of integral values of two or three variables, with the obvious geometric representation by a lattice. Here the author gives a detailed account of his theory of convex figures or solids having a center, in their relations to the lattice. In particular he discusses the following question: How may an infinitude of given congruent central convex solids be placed in parallel positions, such that no two of the solids overlap, such that the centers form a three dimensional lattice, and such that the space not filled by the solids shall be as small as possible? The problem is discussed at length and definitive analytic criteria are deduced (page 104). The text carries the discussion to a final conclusion for spheres and for ellipsoids and applies the results to the immediate derivation of a complete theory of positive ternary quadratic forms, obtaining in particular the results of Gauss and Seeber.

The second half of the text develops the more fundamental parts of the classic theory of algebraic numbers from the author's novel standpoint. The example of cubic number fields $K(\theta)$ is usually chosen to secure simplicity in the geometric interpretations. The integral algebraic numbers of $K(\theta)$ correspond uniquely to the points of a lattice. The existence of a basis for the integral algebraic numbers follows at once geometrically. An inferior limit for the discriminant of a number field is established; except for the field of rational numbers the discriminant exceeds unity. Within the lattice which pictures the totality of integral algebraic numbers there exists (page 156) a lattice whose points picture uniquely the numbers of any given ideal. There follows geometrically the existence of a basis of the ideal. The introduction of lattices not only secures a highly serviceable geometric setting for the algebraic theory, but in connection with related concepts from the author's Geometry of Numbers enables him to establish some fundamental theorems on algebraic numbers which have not been demonstrated without this geometric theory. The final chapter on the approximation of complex quantities by means of numbers of the fields defined by a cube root or a fourth root of unity is the work of Dr. Axer, who also otherwise aided in the publication of the book.

Since the author has given an elementary, entertaining account, both in geometric and arithmetic language, of some important original results as well as the salient features of a classic theory, but presented in a novel manner, his work is deserving of the attention of the very widest circle of readers.

L. E. DICKSON.

Die Lehre von den geometrischen Verwandtschaften. Zweiter Band; die eindeutigen linearen Verwandtschaften zwischen Gebilden zweiter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1908. vi + 346 pp.

THE present volume of Professor Sturm's treatise has about the same programme for two dimensions as the preceding one has for one,* except that non-linear transformations are only considered incidentally. This part is not provided with an index nor a preface, but contains a glossary of 131 technical words besides those found in the preceding one, and the table of contents of all four volumes.

The first chapter (pages 1–218) is concerned with collineation and correlation in the plane, being somewhat similar in scope and

* See the review in the December number of the present volume of the BULLETIN, p. 135.