fields $K(\theta)$ is usually chosen to secure simplicity in the geometric interpretations. The integral algebraic numbers of $K(\theta)$ correspond uniquely to the points of a lattice. The existence of a basis for the integral algebraic numbers follows at once geometrically. An inferior limit for the discriminant of a number field is established; except for the field of rational numbers the discriminant exceeds unity. Within the lattice which pictures the totality of integral algebraic numbers there exists (page 156) a lattice whose points picture uniquely the numbers of any given ideal. There follows geometrically the existence of a basis of the ideal. The introduction of lattices not only secures a highly serviceable geometric setting for the algebraic theory, but in connection with related concepts from the author's Geometry of Numbers enables him to establish some fundamental theorems on algebraic numbers which have not been demonstrated without this geometric theory. The final chapter on the approximation of complex quantities by means of numbers of the fields defined by a cube root or a fourth root of unity is the work of Dr. Axer, who also otherwise aided in the publication of the book.

Since the author has given an elementary, entertaining account, both in geometric and arithmetic language, of some important original results as well as the salient features of a classic theory, but presented in a novel manner, his work is deserving of the attention of the very widest circle of readers.

L. E. DICKSON.

Die Lehre von den geometrischen Verwandtschaften. Zweiter Band; die eindeutigen linearen Verwandtschaften zwischen Gebilden zweiter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1908. vi + 346 pp.

THE present volume of Professor Sturm's treatise has about the same programme for two dimensions as the preceding one has for one,* except that non-linear transformations are only considered incidentally. This part is not provided with an index nor a preface, but contains a glossary of 131 technical words besides those found in the preceding one, and the table of contents of all four volumes.

The first chapter (pages 1–218) is concerned with collineation and correlation in the plane, being somewhat similar in scope and

* See the review in the December number of the present volume of the BULLETIN, p. 135.

treatment to parts of Reye's treatise, volumes two and three. The proof that the vertices of any two quadrangles are sufficient to define projectivity in the plane is identical with that given by Cremona (Elements of projective geometry, Art. 94), but the author acknowledges assistance from a letter from Reye. After the general properties have been considered, metrical properties, particularly vanishing lines and foci are discussed; particular cases, such as affinity, similarity, and congruence are treated at length. In the discussion of two superposed collinear fields, the existence of one real self-corresponding point is assumed. The rest of the discussion is then clear enough, but it seems odd that no proof of this fundamental theorem is given. Homology is given an important place and Hermitian collineations are treated at length. Thus far correlation has hardly been mentioned; it is introduced by duality, showing that a pair of elements may be simply or doubly conjugate. If all elements are doubly conjugate, we have a polar field. The discussion of this part is very detailed; in connection with imaginary elements it is shown that loci exist, all of whose points are imaginary, but arranged in conjugate pairs (Klein's null-partite curves). A large number of constructions of polar fields is given, including the determination of foci, center, conjugate diameters of the invariant conic, and self-conjugate triangle of a pencil or a series of conics. These ideas are now developed in the geometry of the bundle, the procedure being quite similar to that given by Reye, volume 1, chapter 15, but followed by a large number of trigonometric formulas involving relations between conjugate diameters, cyclic planes and principal axes.

Now follows a well-arranged treatment of cyclic collineations. A number of statements are not literally true; *e. g.*, that a cyclic homology of period greater than 2 can not exist (page 162). We gradually see that the author is considering only real elements, but in other articles imaginaries are brought in without warning. The development reminds one of that of Ameseder, but his name is not included among the authors cited. Since the guiding idea is the discussion of forms having two degrees of freedom, quadric and cubic surfaces, space cubic curves and Hirst's complex are legitimate topics of discussion. The polar theory of quadrics is presupposed and a large number of theorems are given without proof, with frequent footnotes giving references to the original memoirs.

Thus far the development has been systematic and comprehensive, and a diligent student could master much of the subject without a teacher, although proofs are much more condensed and the amount of presupposed knowledge much higher than in the preceding volume. The second chapter however (pages 219–346) is concerned almost entirely with problems of enumeration, and is to be regarded as a hand-book, rather than a text. It begins with a short discussion of singular correlations, and uses the results persistently throughout the chapter.

The procedure is as follows: Given the point plane $\Sigma \equiv (A, B, \cdots)$ and the line plane $\Sigma' \equiv (a', b', \cdots)$ correlated in such a way that A has a' for image, B has b', etc., and that B, C, D are collinear, but a', c', d' form a triangle.

The pencil $A(B, C, D)$ and the range $a'(b', c', d')$ are projective, so that corresponding to any line AX will be a definite point X' on a', but other pencils, *e. g.*, $B(A, C, D)$ go into the single point $a'b'$. We can now construct the image of any point or line in either plane. The lines a' and $a \equiv (B, C, D)$ are both singular. This correlation is called axial. If the relation between Σ, Σ' be such that a', c', d' are concurrent, but B, C, D not collinear (dual of the preceding case) the central correlation with the singular points A, A' results.

There are ∞^8 correlations in the plane. If a given point P is associated with a given line p', this counts for two conditions, since two parameters are necessary to fix P. If, however, P is to be on some given line p, without more definitely fixing it, the fact that p, p' are conjugate counts for one condition. Now consider α pairs (p', P), β pairs (P', p), γ pairs (A, A'), δ pairs (a, a') given, wherein these construct the pole of b, and connect it with C' by means of x'; these lines are respectively conjugate to b. Thus a (ν, ν) correspondence between the lines of C' is established, having 2ν coincidences made up of π central collineations and μ having a, b and c' conjugate points, hence $2\nu = \mu + \pi$; and by duality $2\mu = \nu + \lambda$. If the number of correlations μ in the pencil having a given pair of conjugate points be denoted by $\mu(\alpha, \beta, \gamma, \delta)_7$, we may therefore write

$$\mu(\alpha, \beta, \gamma, \delta)_7 = (\alpha, \beta, \gamma + 1, \delta)_8.$$

Finally, if two conditions be suppressed, in $(\alpha, \beta\, \gamma, \delta)_8$ are ∞^1 axial and ∞^1 central collineations. The conjugate points will describe curves, the conjugate lines will envelop others. By use of the chapter on multiple correspondence developed in the first volume it is now possible to determine $(\alpha, \beta, \gamma, \delta)_8$ in each case.

Many of the details are omitted; in order to follow the argument in the later cases of the bundle frequent use must be made of Schubert's Enumerative geometry, Hirst's article in the *Mathematische Annalen*, volume 8, and in particular to the comprehensive memoir of the author in volume 12 of the *Mathematische Annalen*. Contrary to the promise in the preface of the first volume, the treatment is synthetic throughout the volume, thus making the development quite one-sided and restricted.

VIRGIL SNYDER.

Arithmétique Graphique. Les Espaces Arithmétiques, leurs Transformations. Par G. ARNOUX. Paris, Gauthier-Villars, 1908. xii + 84 pp.

THIS book treats of arithmetic spaces which have been defined and studied by the author in two previous books written under the same general title, "Arithmétique Graphique."* The second of these was reviewed by the writer in May, 1907.† The reader is referred to this review for the definition of arithmetic spaces and for a short account of Arnoux's use of them to furnish a graphical representation in the theory of numbers.

In this third book, the author is more interested in the properties of the arithmetic spaces themselves and not so much

* Les espaces arithmétiques hypermagiques, Paris, 1894. Introduction a l'étude des fonctions arithmétiques, Paris, 1906.

† BULLETIN, vol. 13, pp. 402–403.