number of important ideas to which the reader is introduced is surprisingly large, and what is of as great importance, a breadth of outlook pervades the volume that teachers will certainly appreciate.

H. E. HAWKES.

Leçons de Mécanique céleste professeés à la Sorbonne. Par H. POINCARÉ. Paris, Gauthier-Villars. Tome I, 1905, 365 pp.; tome II, 1re partie, 1907, 165 pp.

TISSERAND has written an excellent four volume treatise on celestial mechanics, containing most of the classic contributions to the subject in the spirit and often in the notations of their original authors. Poincaré's Les Méthodes nouvelles de la Mécanique céleste was devoted to establishing, with the rigor of modern mathematical methods, the existence of various kinds of periodic orbits, to determining their properties, to proving the non-existence of new uniform integrals analytic in the masses and the existence of asymptotic solutions, etc. These profound and important researches have almost nothing in common with earlier investigations either in method or subject matter. The Leçons of Poincaré occupy ground between that covered in Tisserand's treatise and in Les Méthodes nouvelles. They are largely expositions of certain parts of standard theories, but the methods of development are those which the author considers as best suited to reaching the desired end regardless of what other writers may have used. It is doubtful if Poincaré could have compelled himself to take the little steps his predecessors have sometimes taken, or to follow their often circuitous paths. At any rate it is fortunate for us that we are able to see how the most penetrating and brilliant mathematician who has written on celestial mechanics reacts on some of our standard theories.

The first volume of the Leçons is devoted to the general theory of planetary perturbations. The first chapter contains an account of canonical equations, their properties, and their transformations. The canonical equations are used throughout the work. The problem of elliptic motion and the equations for the variations of the elliptic elements are developed in these variables, and Lagrange's method of variation of parameters is explained and applied. It is easily shown that the general term in the expression for either a coordinate or an element is of the form

$$\mu^{\alpha} A M t^{m} \cos (\nu t + h),$$

where μ is a small quantity of the order of the masses, A a function of the ratios of the major semiaxes of the orbits, M a power series in the eccentricities and inclinations, and ν a linear homogeneous function of the mean motions. If $m = 0$, $\nu \neq 0$ the term is *periodic*, if $m \neq 0$, $\nu = 0$ the term is purely *secular*, and if $m \neq 0$, $\nu \neq 0$ the term is *mixed secular*. Poincaré calls α the *order* of the term, the degree of the term of lowest degree in the small quantities in M the *degree* of the term, $\alpha - m$ the *rank* of the term, and, if m' represents the exponent of the small divisor arising through integration, $\alpha - m/2 - m'/2$ the *class* of the term. For practical work the importance of a term depends upon its order, degree, rank, and class, and the classical results relating to the stability of the system are attached to relations among these properties. In Chapter V Poincaré has proved in a very simple way, considering the complexity of the subject, that (1) there are no terms of negative rank; (2) there is no mixed secular term of rank zero; and (3) in the expressions for the major axes there is no term of rank zero. These theorems have been known since the time of Poisson for the first two approximations ($\alpha = 1, 2$), and they are recalled here because Poincaré shows they are general.

Only a few of the many results and methods of interest can even be mentioned here, but the conclusions reached in Chapters VIII and IX cannot be passed in silence. These chapters are devoted to a discussion of the terms of rank zero. It is clear that while these terms may be of little importance for small values of t, they are important in determining the configuration of the system for large values of t, certainly if the series in which they occur are convergent. The original differential equations may be written in the form

$$\frac{dx_i}{dt} = \sum_{j=0}^{\infty} B_{ij} \cos (\nu_{ij} t + h_{ij}).$$

Certain of the ν_{ij} are zero. Let the values of B_{ij} for which this is true be $B_{ij}^{(0)}$, and those for which $\nu_{ij} \neq 0$ be $B_{ij}^{(1)}$. Then we have

$$\frac{dx_i}{dt} = \sum B_{ij}^{(0)} \cos h_{ij} + \sum B_{ij}^{(1)} \cos (\nu_{ij} t + h_{ij}).$$

The first terms in the right members give rise to terms of rank zero and order one. Let the terms in $B_{ij}^{(0)}$ of degree one be represented by $B_{ij}^{(00)}$. Lagrange treated the question of stability by taking out these terms and writing

$$\frac{dx_i^{(00)}}{dt} = \sum B_{ij}^{(00)} \cos h_{ij}.$$

Since there is no term in the major axes of rank zero, they are constant so far as these equations are concerned. Making use of this fact, the remaining equations become linear and homogeneous with constant coefficients. Lagrange showed that all the roots of the characteristic equation of this system are purely imaginary, and inferred from this result that the solar system is stable. However, if the process of Lagrange be extended to the terms in $B_{ij}^{(0)}$ of higher degree it is found that secular terms appear in the expressions of the third order.

The result obtained by Poincaré, and given also in Les Méthodes nouvelles, volume II, chapter X, is that the equations

$$\frac{dx_i^{(0)}}{dt} = \sum B_{ij}^{(0)} \cos h_{ij}$$

can be transformed and integrated *formally* so that the $x_i^{(0)}$ involve only trigonometric terms.

In chapter X he shows that the general problem of three bodies may be *formally* integrated in purely trigonometric series, a result established long ago by Newcomb and Tisserand.

Volume II, Part 1, of the Leçons is entirely devoted to the development of the perturbative function. The whole problem is treated with the skill of a master. From the point of view of novelty and mathematical interest chapter XX is the most important. It is devoted to a discussion of the convergence of the series for the coefficients of the development of the perturbative function. The problem is the development of a certain function $F(u, u')$, which is periodic in both u and u' considered separately, in a series of the form

$$F = \sum B_{mm'} e^{i(mu+m'u')}.$$

The $B_{mm'}$ are functions of various of the elements and are expanded as power series in certain of these parameters. They are defined by

$$B_{mm'} = \frac{1}{4\pi^2} \int_{|u|=1} \int_{|u'|=1} F e^{-i(mu+m'u')}\, du du'.$$

The singularities of $B_{mm'}$ as functions of the parameters (let us call them $\beta_1, \cdots, \beta_n$) are found by varying $\beta_1, \cdots, \beta_n$ in F, and finding under what conditions the contours of integration can not be deformed so as to escape passing through a singularity. By a very beautiful discussion the general problem is set up, and the details are worked out in some of the simpler cases.

In conclusion, we may inquire to what class of readers these volumes will most appeal. They certainly can not be digested by American students who are just starting work in the lines to which they are devoted. While Poincaré begins at first principles and works out the results in rather complete detail, the giant strides of the master cross too wide a field of mathematics to be within the grasp of beginners in graduate work. The mathematics would all be readily intelligible to the mature mathematician, but those who were not already somewhat familiar with the subject matter would very often find themselves at a loss to know why the numerous particular artifices were employed. For one having an interest in, and knowledge of, celestial mechanics, this work will be a great source of information and inspiration. In these respects it is surpassed only by the incomparable Méthodes nouvelles, which will, I believe, indirectly revolutionize celestial mechanics even in all its practical details.

F. R. MOULTON.

Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte. Gesamtbericht, herausgegeben von A. GUTZMER. Leipzig, 1908, large 8°, pp. xii + 322.

IN a stately volume there is laid before us the complete report of the Commission appointed in 1904 by the Society of German natural scientists and physicians to examine and report upon various proposed reforms in the teaching of mathematics and and the natural sciences in Germany. The present report contains reprints of the proceedings in Cassel (1903), and in Breslau (1904), which led to the formation of the Commission, and of the proposals submitted by the Commission in Meran (1905), Stuttgart (1906), and Dresden (1907).

The central questions relative to the teaching of mathematics were taken up in the proposals of 1905, which have already been discussed in the BULLETIN * and the later reports deal

* Vol. 12 (1906), pp. 347–352.