$$T = x^6 + y^6 + z^6 + 3x^4y^2 + 3x^4z^2 + 3x^2y^4 + 3x^2z^4 + \sum_{i=1}^{5} c_i y^{6-i} z^i + xyz\phi.$$

Set $y = \lambda z$. Then T becomes

$$t = x^6 + r_2 x^4 z^2 + r_3 x^3 z^3 + \cdot + r_6 z^6,$$

$$r_2 = 3\lambda^2 + A\lambda + 3,\ r_3 = D\lambda^2 + E\lambda,\ r_4 = 3\lambda^4 + F\lambda^3 + M\lambda^2 + G\lambda + 3,$$

$$r_5 = B\lambda^4 + K\lambda^3 + L\lambda^2 + C\lambda,\quad r_6 = \lambda^6 + c_1\lambda^5 + \cdots + c_5\lambda + 1.$$

Now $\tau = x^6 \pm x^3z^3 - z^6$, viz., is of type π, if and only if

$$(21) \qquad r_2 \equiv r_4 \equiv r_5 \equiv 0, \quad r_3^2 \equiv 1, \quad r_6 \equiv -1 \pmod 7;$$

while τ is a perfect cube if and only if

$$(22) \qquad r_3 \equiv r_5 \equiv 0, \quad r_4 \equiv 5r_2^2, \quad r_6 \equiv 6r_2^3 \pmod 7.$$

Since $r_2 r_3 \equiv 0$ for every λ, $D \equiv E \equiv 0$. Hence (21) is excluded, so that (22) must hold for every λ. We may therefore remove the term y^5z from T and proceed as in § 10. Or we may proceed with (22) and show that

$$T \equiv (x^2 + y^2 + z^2 - 2Ayz)^3.$$

12. The theorem that there exists no sextic on three or more variables which represents only cubes in a field of order $p^n = 3k + 1$ has now been established for $p^n < 31$, $p^n = 2^n$, and $p^n = 11^2$. Its truth for all values of p^n has been proved, subject to the validity of the conjectured theorem of § 9 on binary sextics.

THE UNIVERSITY OF CHICAGO,
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NOTE ON LÜROTH'S TYPE OF PLANE QUARTIC CURVES.

BY PROFESSOR H. S. WHITE AND MISS KATE G. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

ONE of the stock examples of the fallacy of constant counting is the equation of a plane quartic, whose fourteen constants equal in number those apparent in the sum of five fourth powers of linear expressions

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4.$$

Clebsch showed why this is not a normal form for all ternary quartics; it has by virtue of its form an infinite system of linear transformations into itself, and so contains actually but thirteen constants. The curve has an apolar, *i. e.*, a triply apolar conic, and the sides of its polar quinquilaterals form a simple quintic involution of tangents to this conic. This special type of quartic is called the Clebschian.

Lüroth's quartic, closely related to this, is an equally alluring source of fallacy. It contains inscribed complete quinquilaterals, all ten vertices lying on the curve. At first sight one would say that every quartic might be of this type. For we may take three variable points, A, B, C on the curve, join them by straight lines, and so have a triangle with three degrees of freedom. Each side cuts the curve in two more points, say L_1 and L_2 on BC, M_1 and M_2 on CA, and N_1, N_2 on the third side. One condition is imposed on the variable points by requiring three points L_1, M_1, N_1, to be collinear; let their line meet the quartic again in P. Finally we should expect two more conditions to be imposed by requiring the four points L_2, M_2, N_2, and P to be collinear; three conditions in all upon the three variable points A, B, and C. If so, the construction might be possible for a finite number of positions, and for each position the five lines of the diagram would constitute an inscribed complete quinquilateral. Unfortunately for the argument, one such inscribed quinquilateral in a quartic curve proves the existence of an infinite number of others instead of a finite number, hence our hasty inference that this type is the most general quartic is nugatory. Lüroth's quartic and the Clebschian are related by the fact that all the inscribed quinquilaterals of the one are the polar quinquilaterals of the other.

Lüroth's quartic is evidently a projective covariant curve of the Clebschian, and has been studied as such. If a certain invariant B (the catalecticant) of the ternary quartic c_x^4 or c vanishes, then c is of the Clebsch type, and its covariant S gives the equation $S=0$ of the corresponding Lüroth quartic

$$S = (cc'c'')(cc'c''')(cc''c''')(c'c''c''')c_x c'_x c''_x c'''_x = 0.$$

The direct and unique connection between the two is thus completely exhibited when the Clebschian is given as the starting point. From it both the Lürothian and the related conic are found as rational covariants,* and the condition which

* See statements and references by Coble in *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 65–85.

must be satisfied is expressed by a rational invariant equation. But suppose the Lürothian equation to be given, $l_x^4 = 0$. Could we then find rationally in its coefficients the Clebschian, the conic, and the characteristic invariant condition for the Lürothian type? This has not been announced, yet apparently it should be possible. We wish here to outline a possible answer, although the result is not so simple as to be entirely satisfactory.

Let the Lürothian, Clebschian, and conic be represented by the equations

$$l_x^4 = 0, \quad c_x^4 = 0, \quad \text{and} \quad u_\alpha^2 = 0,$$

respectively. These are connected by a relation that seems not to have been employed hitherto. It is well known that tangents to the conic are cut by the two quartics in apolar quartettes of points. But it is further true that *these tangents are the only lines whose intersections with the quartics are mutually apolar.* The envelope of such lines is of the fourth class,

$$(clu)^4 = 0,$$

and if the conic forms part of this envelope, a second conic must form the complementary part. That this complementary part is the same conic repeated can be verified easily from the normal forms of the equations

$$l_x^4 = \sum_1^5 (x_2 x_3 x_4 x_5) \quad \text{and} \quad c_x^4 = \sum_1^5 x_1^4.$$

We have therefore the formal equation, identical in the u's,

$$(1) \qquad (lcu)^4 \equiv (u_\alpha^2)^2.$$

The triple apolarity of conic and Clebschian is expressed by the identity in the variables x_1, x_2, x_3,

$$(2) \qquad c_\alpha^2 c_x^2 \equiv 0.$$

The identity (1) yields 15 equations linear in the quantities $c_{ijk} (= c_1^i c_2^j c_3^k)$; from these we find the Clebschian explicitly as a fifteen-rowed matrix bordered with one column of the second degree in the coefficients of u_α^2, and one row quartic in variables x_1, x_2, x_3. Call the determinants $|L|$, and the bordered matrix

$$\begin{vmatrix} L & (\alpha)^2 \\ (x)^4 & 0 \end{vmatrix},$$

and we write

$$(3) \qquad | L | \cdot c_x^4 \equiv \begin{vmatrix} L & (\alpha)^2 \\ (x)^4 & 0 \end{vmatrix}.$$

Hence, applying condition (2), we find what may be symbolized thus

$$(4) \qquad \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix} \equiv 0 \text{ for all values of the } x_1, x_2, x_3.$$

This contains six cubic equations for the six coefficients of the conic. *If these can be satisfied simultaneously, the quartic* l_x^4 *is of Lüroth's type; equation (3) gives the Clebschian in terms of the Lürothian and the conic; and the common solution*

$$(\alpha_{11}, \alpha_{12}, \cdots, \alpha_{33})$$

of the six equations (4) is the conic envelope.

We may state the relation of things more concisely by using the terminology of space of five dimensions, treating the coefficients α_{ik} of the conic as homogeneous coordinates in such a space. Replace the powers and products of x_1, x_2, x_3 in equation (4) by a second set of coordinates β_{ik} cogredient with the α_{ik}; then equation (4) is that of the first (cubic) polar of a point β in the five-space, with respect to a quartic locus or four-spread in that space, whose equation may be denoted by

$$(5) \qquad \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix} = 0,$$

since

$$\begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix}_{x_i x_k = \beta_{ik}} \equiv \tfrac{1}{2}\Sigma\left(\beta_{ik}\frac{\partial}{\partial \alpha_{ik}}\right) \cdot \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix}.$$

This first polar equated to zero, identically in the β_{ik}, is the requirement that the quartic four-spread (5) shall have an ordinary double point or node, viz., at the point (α). Therefore, *the problem of finding the invariant condition satisfied by the Lürothian is equivalent to the problem of finding the discriminant of a certain quartic* (5) *homogeneous in six variables. If that discriminant vanishes, the determination of the covariant conic* α *is effected by determining the coordinates of the node of that quartic four-spread; and from this conic and the Lürothian the Clebschian* c_x^4 *is then found uniquely by equation* (3).

That this solution shall not be illusory, of course the determinant $|L|$ must not vanish. For a particular quartic, one containing only even powers of x_1, x_2, x_3, this test is made very easily; and it is of interest to record that then this determinant $|L|$ factors into two rational expressions of degrees 6 and 9 respectively, neither of them containing Salmon's invariant B as a factor. This shows that $|L|$ is not zero for a quartic L of the Clebschian type. But also for the Lüroth type of quartic it does not vanish. This we have proved with some labor by taking l_x^4 in the canonical form

$$l_x^4 = Ax_2x_3x_4x_5 + Bx_1x_3x_4x_5 + \cdots + Ex_1x_2x_3x_4,$$

with the two linear relations

$$x_4 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3, \quad x_5 = \beta_1x_1 + \beta_2x_2 + \beta_3x_3,$$

and computing a sample term of $|L|$. We selected the one containing the quantities C and β_3 to the highest possible powers, namely the term in

$$A^2B^2C^{11}\alpha_1^7\alpha_2^5\alpha_3^3\beta_2^2\beta_3^{13}.$$

Such a term can occur in only 21 terms of the expanded determinant $|L|$, and carefully repeated computation gives its coefficient as -8. While we should prefer some simpler verification of this point, yet we regard this trial as sufficient.

Owing to the high degree of the discriminant of a senary quartic, it seems at least likely that for a quartic of the particular type $\left|\begin{smallmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{smallmatrix}\right|$ the discriminant might prove to be an exact power of some invariant of lower order in the given coefficients l_{ik}. Such a reduction we have proved to exist, so far, only in certain analogous forms in fewer variables.

When the Lürothian is regarded as derived from a quintic involution among the tangents of a conic, F. Meyer has shown precisely how to determine the (4, 4) correspondence among those tangents that represents the Clebschian. This mode of determination he extends to forms of any even order. An involutional sextic curve is always uniquely associated with another sextic which is apolar to the septilaterals inscribed in the former. The relation between such a pair of sextics (octics, etc.) in ternary variables can be expressed by an obvious generalization of the identities (3) and (4). For example if $l_x^6 = 0$ and $c_x^6 = 0$ are two such sextics, and $u_\alpha^2 = 0$ the conic carrying the related involution, then the envelope

$$(clu)^6 = 0$$

is the triply counted conic: $(clu)^6 \equiv \rho \cdot (u_\alpha^2)^3$. So for the similarly related pairs of higher order.

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CANTOR'S HISTORY OF MATHEMATICS.

Vorlesungen über Geschichte der Mathematik. By MORITZ CANTOR. Vierter Band. Von 1759–1799. Leipzig, Teubner, 1908. 8vo. vi + 1113 pp. 32 Marks.

IT is a long period that Moritz Cantor looks back upon to the time when he made his first noteworthy contribution to the history of mathematics. He was little more than a boy when he published his Inaugural-Dissertation "Ueber ein weniger gebräuchliches Coördinaten-System" in 1851,—only twenty-two years old; and it was only five years later that he entered upon his life work as historian of mathematics, by the publication of his paper "Ueber die Einführung unserer gegenwärtigen Ziffern in Europa." He was thirty-four when his first noteworthy treatise on the history of mathematics appeared, the "Mathematische Beiträge zum Kulturleben der Völker" (Halle, 1863). And next August he will be eighty years old, with over a dozen books to his credit, with hundreds of memoirs, with a notable record as editor of the *Zeitschrift für Mathematik und Physik* and the *Abhandlungen zur Geschichte der Mathematik*, and with academic honors and governmental recognition commensurate with the great work that he has accomplished.

For a man of nearly eighty to undertake a work of the magnitude of this fourth volume of the Geschichte would seem foolhardy, were he not such a man as Professor Cantor. When the third volume was completed in 1898 it seemed proper that it should be called the last, as was the case, and under all the circumstances any man might have been well content to say his *nunc dimittis*. That Professor Cantor was not thus content is a cause of gratification to all who are interested in the history of mathematics, and a worthy lesson to all who feel that three-score years mark the bounds of man's working life.

The present volume is not the work of Cantor himself in the