TAUTOCHRONES AND BRACHISTOCHRONES.

BY PROFESSOR EDWARD KASNER.

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IN the simplest case of a particle acted upon by gravity, the tautochrone problem, solved by Huygens in 1673, and the brachistochrone problem, solved by Jean Bernoulli in 1697, give rise to the same curves, namely, cycloids with horizontal bases and concavity upwards. In the case of a general field of force the two problems lead to distinct systems of curves. Their differential equations, each of the third order, are given in the first section of this note. In § 2 it is shown that the only force besides gravity for which the two systems coincide is the central force varying directly as the distance from the origin. If the force generating the brachistochrones is not required to be the same as that generating the tautochrones, then a third case of duplication is possible (§ 4). Incidentally, a class of forces involving eight parameters and related to infinitesimal collineations presents itself (§ 3); they are the only fields of force for which every straight line of the plane is a tautochrone.

§ 1. *General Equations.*

We consider a particle of unit mass moving in the plane under a force whose rectangular components are $\phi(x, y)$, $\psi(x, y)$. With reference to an arbitrary curve the normal and tangential components are

$$(1) \qquad N = \frac{\psi - y'\phi}{\sqrt{1 + y'^2}}, \quad T = \frac{\phi + y'\psi}{\sqrt{1 + y'^2}}.$$

The condition for a tautochrone is that the motion along the curve be harmonic, that is,

$$(2) \qquad T = \kappa(s - s_0),$$

where κ is a constant* and $s - s_0$ denotes the arc reckoned

* For an *actual* tautochrone κ must be negative. The differential equation (3) applies also when κ is positive. Such curves may be termed *virtual* tautochrones. They are the actual tautochrones of the reversed force. A similar remark applies to trajectories, brachistochrones, and catenaries.

from a fixed point of the curve, the center of the tautochronous motion. Differentiating twice with respect to s, we have, as the equation of all the tautochrones,

$$T_{ss} = 0. \tag{3}$$

The brachistochrones may be found most readily from Euler's theorem on pressure, which gives

$$\frac{v^2}{r} = -N, \tag{4}$$

where v is the speed and r the radius of curvature.

Differentiating both sides with respect to s, and remembering that

$$vv_s = \frac{dv}{dt} = T, \tag{5}$$

we find the equation of the family of brachistochrones to be *

$$2T + rN_s + Nr_s = 0. \tag{6}$$

Equations (3) and (6) when expanded are seen to involve third derivatives of y with respect to x, so that each defines a triply infinite system of curves.

To obtain the results in more explicit form we introduce an auxiliary vector, completely defined by the given field of force, namely, the space derivative of the force (considered as a vector). The rectangular components of the new vector are

$$\phi_s = \frac{\phi_x + y'\phi_y}{\sqrt{1+y'^2}}, \quad \psi_s = \frac{\psi_x + y'\psi_y}{\sqrt{1+y'^2}}, \tag{7}$$

and its normal and tangential components are

$$\begin{aligned} \mathfrak{N} &= \frac{\psi_s - y'\phi_s}{\sqrt{1+y'^2}} = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{1+y'^2}, \\ \mathfrak{T} &= \frac{\phi_s + y'\psi_s}{\sqrt{1+y'^2}} = \frac{\phi_x + (\phi_y + \psi_x)y' + \psi_y y'^2}{1+y'^2}. \end{aligned} \tag{8}$$

* This result, like Euler's theorem, applies only to conservative forces. Apparently the theory of brachistochrones for non-conservative fields has not been investigated. The system in this case consists in fact of ∞^4 instead of merely ∞^3 curves. Since the tautochrone system always contains ∞^3 curves, such forces need not be considered in the problems of § 2 and § 4.

The following noteworthy relations may be derived without difficulty:

$$(9) \qquad N_s = \mathfrak{N} - \frac{T}{r}, \qquad T_s = \mathfrak{T} + \frac{N}{r}.$$

We require finally the space derivatives of $\mathfrak{N}$ and $\mathfrak{T}$, which may be written in the form

$$(10) \qquad \mathfrak{N}_s = \mathfrak{N}_1 + \frac{\mathfrak{N}_2}{r}, \qquad \mathfrak{T}_s = \mathfrak{T}_1 + \frac{\mathfrak{T}_2}{r},$$

where

$$(11) \qquad \begin{aligned} \mathfrak{N}_1 &= \frac{\psi_{xx} + (2\psi_{xy} - \phi_{xx})y' + (\psi_{yy} - 2\phi_{xy})y'^2 - \phi_{yy}y'^3}{(1 + y'^2)^{\frac{3}{2}}}, \\ \mathfrak{N}_2 &= \frac{\psi_y - \phi_x - 2(\phi_y + \psi_x)y' + (\phi_x - \psi_y)y'^2}{1 + y'^2}, \\ \mathfrak{T}_1 &= \frac{\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3}{(1 + y'^2)^{\frac{3}{2}}}, \\ \mathfrak{T}_2 &= \frac{\phi_y + \psi_x + 2(\psi_y - \phi_x)y' - (\phi_y + \psi_x)y'^2}{1 + y'^2}. \end{aligned}$$

The functions ϕ, ψ depend only on the position of the particle; the auxiliary functions N, T, $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$, defined above, depend also upon the direction of motion; finally, N_s, T_s, $\mathfrak{N}_s$, $\mathfrak{T}_s$ depend upon the curvature of the path.

Making use of (9) and (10), we may reduce our equations (3) and (6) to explicit form, and obtain this result:

The general equation of the system of tautochrones is

$$(\text{I}) \qquad Nr_s = \mathfrak{T}_1 r^2 + (\mathfrak{T}_2 + \mathfrak{N})r - T,$$

while that of the system of brachistochrones is

$$(\text{II}) \qquad Nr_s = -\mathfrak{N}r - T.$$

From these equations we may deduce the general geometric properties of the two systems. We note merely that for tautochrones r_s (the rate of variation of the radius of curvature per unit of arc) is a quadratic function of r, while for brachistochrones it is a linear function of r. The results (I) and (II) are of course differential equations of the third order with respect to y as a function of x. The usual notation may be obtained by substituting

$$(12) \qquad r = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}, \quad r_s = \frac{3y'y''^2 - (1+y'^2)y'''}{y''^2},$$

but the above intrinsic forms are more convenient in the following.

We note in passing that for any field of force the equation

$$(13) \qquad Nr_s = -\mathfrak{N}r + mT,$$

involving an arbitrary parameter m, includes the following four distinct systems of curves of physical interest: brachistochrones when $m = -1$, trajectories when $m = 3$, catenaries when $m = 2$, and velocity curves when $m = 1$. For all values of m the curves may be obtained by requiring that the pressure shall vary directly as the normal component of the external force. No cases of duplication, beyond those already mentioned, are found by comparing these systems with tautochrones or with one another.

§ 2. *The Duplication Problem for a Single Field.*

We proceed to find the forces for which the brachistochrone system coincides with the tautochrone system. The conditions expressing the equivalence of equations (I) and (II) are

$$(14) \qquad \mathfrak{T}_1 = 0,$$

$$(15) \qquad \mathfrak{T}_2 + 2\mathfrak{N} = 0.$$

Expanding the latter, we find

$$(15') \qquad \phi_y + 3\psi_x = 0, \quad \psi_y - \phi_x = 0, \quad 3\phi_y + \psi_x = 0.$$

The solutions are easily found to be

$$(16) \qquad \phi = cx + a, \quad \psi = cy + b,$$

where a, b, c are arbitrary constants. No new restrictions are imposed by (14). By a simple change of axes and scale the field (16) may be written in one of these two canonical forms,

$$(17) \qquad \phi = 0, \quad \psi = 1,$$

$$(18) \qquad \phi = x, \quad \psi = y.$$

The only types of force for which the tautochrones coincide with the brachistochrones are represented by (17) *and* (18). *In the first type the force is constant in direction and intensity* (*gravity*); *in the second type the force is central and varies directly as the distance from the center* (*elastic law*).

The system of curves in the first type consists of cycloids with horizontal bases; in the second type the curves are hypocycloids and epicycloids, including as limiting cases equiangular spirals, all having the origin as center.

§ 3. *A Class of Forces Related to Collineations.*

We now determine those forces for which the right hand member of the general tautochrone equation (I) is linear in r.* The condition for this is $\mathfrak{T}_1 = 0$, which decomposes on expansion into the set of partial differential equations

$$(19) \qquad \begin{aligned} \phi_{xx} = 0, \quad 2\phi_{xy} + \psi_{xx} = 0, \\ \phi_{yy} + 2\psi_{xy} = 0, \quad \psi_{yy} = 0. \end{aligned}$$

The integration of this set may be carried out without difficulty; but the calculation may be avoided by noticing a simple connection with the equations arising in the determination of infinitesimal collineations. If an infinitesimal transformation, represented symbolically by

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y},$$

is to be projective, then the well known conditions are

$$\begin{aligned} \eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \\ \eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \end{aligned}$$

This is converted into our set (17) by replacing ξ by ψ and η by $-\phi$. Hence the solutions of (17) are

$$(20) \qquad \begin{aligned} -\phi = b + ex + gy + hxy + ky^2, \\ \psi = a + cx + dy + hx^2 + kxy, \end{aligned}$$

involving eight arbitrary constants, the notation being taken in agreement with Lie-Scheffers, Continuierliche Gruppen, page 24. Our result is then:

* The linear form is equivalent to property I of a system of dynamical trajectories. See *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 405.

If a system of tautochrones is to have the property that r_s *(the rate of variation of the radius of curvature per unit of arc) is a linear function of* r *(the radius of curvature), then the field of force must be of the class* (20). *These fields are characterized by the fact that the related infinitesimal transformation*

$$\psi\frac{\partial}{\partial x}-\phi\frac{\partial}{\partial y} \tag{21}$$

is a collineation.

It follows that the lines of force,

$$\frac{dy}{dx}=\frac{\psi}{\phi}, \tag{22}$$

have for orthogonal trajectories a system of anharmonic curves (W curves). Of course this property is not characteristic, since it describes only the direction of the force, not its intensity.

The forces (20) arise most concretely in response to this problem: In what cases will every straight line be a possible tautochrone? *

For a straight line we have $r=\infty$, and $r=0$. If (I) is to be satisfied by these values, then $\mathfrak{T}_1$, the coefficient of r^2, must vanish. The same result is obtained by reducing (I) to the form

$$y'''=\mathfrak{T}_1+y''f_1(x,y,y')+y''^2f_2(x,y,y'), \tag{23}$$

and imposing the condition that $y''=0$ shall be a particular integral.

The class of forces (20) *is thus completely characterized by the fact that every straight line of the plane is a possible tautochrone.*

If the force (20) is required to be conservative it is found that the coefficients must satisfy the relations

$$h=0,\quad k=0,\quad c+g=0. \tag{24}$$

The corresponding collineation then becomes an affine transformation leaving areas invariant. The components of the force are of the linear form

$$\phi=-b-ex+cy,\quad \psi=a+cx+dy$$

and the work function is

* Mr. Reddick has shown that in space of three dimensions this problem leads to a class of forces involving twenty parameters; the relation to collineations is therefore peculiar to the plane.

(25) $$W = -bx + ay + cxy - \tfrac{1}{2}ex^2 + \tfrac{1}{2}dy^2.$$

The only conservative forces for which every straight line is a tautochrone are those in which the potential is a quadratic function of x, y.

The lines of force are then the orthogonals of a system of homothetic conics. Such forces are of importance in connection, for example, with the general theory of motion about a position of equilibrium.

§ 4. *The Duplication Problem for Two Fields.*

When will the system of ∞^3 tautochrones connected with one field of force coincide with the system of ∞^3 brachistochrones connected with a second field? We denote the components of the first force by ϕ, ψ, and those of the second force by ϕ^*, ψ^*, so that the differential equations of the two systems considered are

(26) $$Nr_s = \mathfrak{T}_1 r + (\mathfrak{T}_2 + \mathfrak{N})r - T,$$

(27) $$N^* r_s = -\mathfrak{N}^* r - T^*.$$

We observe that a necessary condition for equivalence is $\mathfrak{T}_1 = 0$; hence the first force must be of the type (20). The additional conditions to be satisfied are

(28), (29) $$\frac{T^*}{N^*} = \frac{T}{N}, \quad \frac{\mathfrak{T}_2 + \mathfrak{N}}{N} = -\frac{\mathfrak{N}^*}{N^*}.$$

From the first of these, we find that the ratio of ψ^* to ϕ^* must be equal to the ratio of ψ to ϕ. We may therefore write

(30) $$\phi^* = \rho\phi, \quad \psi^* = \rho\psi,$$

where ρ is an unknown function of x, y. A short calculation shows that, in consequence of (30),

(31) $$\frac{\mathfrak{N}^*}{N^*} = \frac{\mathfrak{N}}{N} + \frac{\sigma_x + y'\sigma_y}{\sqrt{1 + y'^2}} \qquad (\sigma = \log \rho).$$

Equation (29) may then be written in the form

(32) $$\mathfrak{T}_2 + 2\mathfrak{N} + N\sigma_s = 0.$$

Expanding this in powers of y', making use of the known form (20) of ϕ and ψ, and equating coefficients to zero, we find

$$(32') \qquad \begin{aligned} 3c - g + 5hx + ky + \psi\sigma_x &= 0, \\ 4(d + e + kx + hy) + \psi\sigma_y - \phi\sigma_x &= 0, \\ 3g - c + hx + 5ky - \phi\sigma_y &= 0. \end{aligned}$$

The unknowns are now the eight coefficients involved in (20) and the function σ. Elimination of σ_x and σ_y gives

$$(33) \qquad \begin{aligned} \phi^2(3g - c + hx + 5ky) + 4\phi\psi(d + e + kx + hy) \\ + \psi^2(3c - g + 5hx + ky) = 0. \end{aligned}$$

By means of the coefficients of the highest powers here involved, namely, x^5 and y^5, we find that h and k must vanish. The last equation thus simplifies to

$$(33') \qquad (3g - c)\phi^2 + 4(d + e)\phi\psi + (3c - g)\psi^2 = 0,$$

where

$$(34) \qquad \phi = -b - ex - gy, \quad \psi = a + cx + dy.$$

We now divide the discussion into two cases:

Case 1°. If the coefficients $3g - c$, $4(d + e)$, $3c - g$ appearing in (33′) all vanish, then $c = 0$, $g = 0$, $d = -e$, and we obtain the form (16) of § 2. In fact, under these conditions, equations (32′) show that both σ_x and σ_y will vanish, that is, σ and hence ρ will be a constant, that is, the two fields will coincide.

Case 2°. Under the contrary assumption, we see from (33′) that the ratio of ψ to ϕ must be constant. This means that the force has a constant direction, which we may without loss of generality assume to be the direction of the y-axis. The component ϕ will then vanish, so that, from (34),

$$b = e = g = 0.$$

Equation (33′) then gives $3g - c = 0$, so that c also must vanish. Our force thus becomes

$$\phi = 0, \quad \psi = a + dy.$$

If d vanishes, this becomes the type (17). Otherwise, it may, by a change of axes and scale, be reduced to the form

$$(35) \qquad \phi = 0, \quad \psi = y.$$

Substituting these values in (32′) we find that the equations for the unknown function σ are consistent and give

$$\sigma = -4 \log y,$$

so that ρ, the factor for converting the first field into the second field, is y^{-4}. Hence the second field is

$$(36) \qquad \phi^* = 0, \quad \psi^* = y^{-3}.$$

In (35) the force varies directly as the distance from the x axis, while in (36) the force varies as the inverse cube of that distance.

The only cases in which a system of tautochrones is also a system of brachistochrones are these three:

1°. *The tautochrones and the brachistochrones of the uniform field $\phi = 0$, $\psi = 1$ coincide.*

2°. *The same is true of the elastic field $\phi = x$, $\psi = y$.*

3°. *The tautochrones of the field $\phi = 0$, $\psi = y$ coincide with the brachistochrones of the field $\phi^* = 0$, $\psi^* = y^{-3}$.*

All these fields are conservative.

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DEGENERATE PENCILS OF QUADRICS CONNECTED WITH $\Gamma^{n+2}_{n+4,n}$ CONFIGURATIONS.

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IN a previous paper * the author discussed a certain pencil of quadric spreads associated with the configuration $\Gamma^{n+2}_{n+4,n}$ in space of n dimensions. The $\Gamma^{n+2}_{n+4,n}$ contains $n+4$ configurations $\Gamma^{n+1}_{n+3,n}$, and with each of these is associated a quadric spread with respect to which its points and S_{n-1}'s are poles and polars. In the case of a *proper* $\Gamma^{n+1}_{n+3,n}$, *i. e.*, one whose points, lines, planes, $\cdots$, and S_{n-1}'s are all distinct, it is evident that the associated quadric spread cannot degenerate into a cone.† Hence, for a proper $\Gamma^{n+2}_{n+4,n}$, the individual spreads of the associated pencil cannot be degenerate; but the question naturally arises as to whether the pencil itself, or the quartic $(n-2)$-way spread through which the quadrics all pass, may be degenerate. It is the object of this paper to answer this question for the cases $n = 1$, 2, and 3.

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* "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$." *Amer. Jour. of Mathematics*, vol. 31, pp. 1–17 (January, 1909).

† That is, if the quadric spread be represented by a quadratic equation in $n+1$ homogeneous variables, the discriminant of this equation must not vanish.