so that ρ, the factor for converting the first field into the second field, is y^{-4}. Hence the second field is

$$(36) \qquad \phi^* = 0, \quad \psi^* = y^{-3}.$$

In (35) the force varies directly as the distance from the x axis, while in (36) the force varies as the inverse cube of that distance.

The only cases in which a system of tautochrones is also a system of brachistochrones are these three:

1°. *The tautochrones and the brachistochrones of the uniform field $\phi = 0$, $\psi = 1$ coincide.*

2°. *The same is true of the elastic field $\phi = x$, $\psi = y$.*

3°. *The tautochrones of the field $\phi = 0$, $\psi = y$ coincide with the brachistochrones of the field $\phi^* = 0$, $\psi^* = y^{-3}$.*

All these fields are conservative.

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DEGENERATE PENCILS OF QUADRICS CONNECTED WITH $\Gamma^{n+2}_{n+4,n}$ CONFIGURATIONS.

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(Read before the American Mathematical Society, December 31, 1908.)

IN a previous paper * the author discussed a certain pencil of quadric spreads associated with the configuration $\Gamma^{n+2}_{n+4,n}$ in space of n dimensions. The $\Gamma^{n+2}_{n+4,n}$ contains $n+4$ configurations $\Gamma^{n+1}_{n+3,n}$, and with each of these is associated a quadric spread with respect to which its points and S_{n-1}'s are poles and polars. In the case of a *proper* $\Gamma^{n+1}_{n+3,n}$, *i. e.*, one whose points, lines, planes, $\cdots$, and S_{n-1}'s are all distinct, it is evident that the associated quadric spread cannot degenerate into a cone.† Hence, for a proper $\Gamma^{n+2}_{n+4,n}$, the individual spreads of the associated pencil cannot be degenerate; but the question naturally arises as to whether the pencil itself, or the quartic $(n-2)$-way spread through which the quadrics all pass, may be degenerate. It is the object of this paper to answer this question for the cases $n = 1$, 2, and 3.

* "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$." *Amer. Jour. of Mathematics*, vol. 31, pp. 1–17 (January, 1909).

† That is, if the quadric spread be represented by a quadratic equation in $n+1$ homogeneous variables, the discriminant of this equation must not vanish.

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We will consider the three-dimensional case first, as affording the best illustration of the method of treatment. We have a pencil of seven quadric surfaces associated with a $\Gamma^5_{7,3}$ configuration. It was shown in the author's previous paper that the pencil of quadrics could be first chosen arbitrarily, and that the configuration was then formally determined. The analytic procedure there described * may be used here. We may choose a pencil of quadric surfaces of any one of the thirteen well-known types,† and see whether the configuration then determined is or is not degenerate.

For example, consider a pencil of quadrics for which the common quartic curve consists of a conic and two lines not intersecting on the conic (type 8 in the Clebsch-Lindemann classification). Seven surfaces of such a pencil may be represented with perfect generality by the equations

$$x_2x_3 + x_3x_4 + x_4x_2 + \lambda_i x_1 x_4 = 0 \quad (i = 1, 2, \cdots, 7).$$

Let the point 123 of the configuration be a_1, a_2, a_3. Then the plane 14567 will be

$$\lambda_1 a_4 x_1 + (a_3 + a_4)x_2 + (a_2 + a_4)x_3 + (\lambda_1 a_1 + a_2 + a_3)x_4 = 0;$$

the point 234 will be

$$\lambda_1\lambda_4 a_1 + 2(\lambda_1 - \lambda_4)a_4,\ \lambda_4^2 a_2 + \lambda_4(\lambda_4 - \lambda_1)a_4,$$
$$\lambda_4^2 a_3 + \lambda_4(\lambda_4 - \lambda_1)a_4,\ \lambda_1\lambda_4 a_4;$$

the point 345 will be

$$\lambda_2\lambda_5\lambda_4\lambda_1 a_1 + 2\,[\lambda_2\lambda_5(\lambda_1 - \lambda_4) + \lambda_1\lambda_4(\lambda_2 - \lambda_5)]\,a_4,$$
$$\lambda_4^2\lambda_5^2 a_2 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_4^2\lambda_5^2 a_3 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_2\lambda_5\lambda_1\lambda_4 a_4;$$

and the plane 12367 will be

$$\Sigma_2 S_2 a_4 x_1 + \Sigma_2^2(a_3 + a_4)x_2 + \Sigma_2^2(a_2 + a_4)x_3$$
$$+ [\Sigma_2 S_1 a_1 + \Sigma_2^2(a_2 + a_3) + 2(\Sigma_2^2 + S_3\Sigma_1 - S_2\Sigma_2)a_4]x_4 = 0,$$

where the Σ's and S's are the ordinary symmetric functions respectively of λ_4, λ_5 and λ_1, λ_2, λ_3. Since the point 123 must lie in the plane 12367, we have the relation

* Loc. cit., p. 5.

† Clebsch-Lindemann, Vorlesungen über Geometrie, vol. II, p. 215.

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$$\Sigma_2 S_3 a_1 a_4 + \Sigma_2^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1 S_3 - \Sigma_2 S_2)a_4^2 = 0.$$

Similarly, the coordinates of the point 123 must satisfy the conditions

$$\Sigma_2' S_3 a_1 a_4 + \Sigma_2'^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1' S_3 - \Sigma_2' S_2)a_4^2 = 0$$

and

$$\Sigma_2'' S_3 a_1 a_4 + \Sigma_2''^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1'' S_3 - \Sigma_2'' S_2)a_4^2 = 0,$$

where the Σ''s and Σ'''s are symmetric functions respectively of λ_5, λ_6 and λ_6, λ_7.

Each of the 35 points of our configuration must satisfy such a set of three equations. These equations are linear and homogeneous in the quantities $a_1 a_4$, $a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4$, and a_4^2. The determinant of the coefficients cannot vanish; for it reduces to

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_5^2 \lambda_6^2 (\lambda_6 - \lambda_4)(\lambda_4 - \lambda_7)(\lambda_7 - \lambda_5)$$

and no λ can be zero (for this would give a cone) and no two λ's can be equal. Hence the conditions could only be satisfied if

$$a_1 a_4 = 0, \quad a_4^2 + a_2 a_3 + a_3 a_4 + a_2 a_4 = 0, \quad a_4^2 = 0.$$

But this would make the point 123 lie on one of the two lines which make up part of the quartic curve, and *the same would be true for all the points of the configuration.* Hence we cannot have a proper configuration in this case.

By a similar treatment in the other cases, we obtain the following results:

Proper configurations exist for the types of pencils 1–5, and do not exist for types 6–13. For the general case, type 1, there are eight configurations for any given pencil of seven quadrics;* but for types 2–5 there are fewer configurations for a given pencil, because some of the solutions are absorbed by the special points or lines.

We find a more surprising situation, perhaps, in the plane case. Here we have five types of pencils of conics:

1. Four distinct points of intersection;
2. Two distinct points of intersection, and contact of first order at a third point;
3. Contact of first order at two distinct points;

* Author's paper, loc. cit., p. 6.

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4. One point of intersection, and contact of second order at a second point;
5. Contact of third order.

Proper configurations exist for types 1 and 4, and do not exist for types 2, 3, and 5. For a given pencil of six conics there are four configurations for type 1, but only one for type 4.

For the one-dimensional case, we have the configuration $\Gamma^3_{5,1}$ on a line, and the pencil of quadrics is simply five pairs of points in involution.* We have two types, viz., the fixed points of the involution are distinct or they are coincident. Proper configurations exist for both cases. Given the five point pairs, there are two configurations determined in the first case, but only one in the second. This one-dimensional case is readily seen if the whole figure is projected upon a conic.

ON THE USE OF n-FOLD RIEMANN SPACES IN APPLIED MATHEMATICS.

BY PROFESSOR JAMES MCMAHON.

THE object of this article is to show that the conception of a Riemann surface has important physical bearings, and to indicate in a general way what kind of physical problems have been solved or may be solvable by the use of such n-fold surfaces or analogous manifold regions in three dimensions. The most recent work in this line constitutes the highest point yet reached in the application of modern function theory to physical problems. It is very noteworthy that a theory which was developed by following out purely intellectual relations, without any reference to the world of sense, should afterwards find unexpected applications and correspondences in the physical universe.

The conceptions of multiform functions, and of multiple spaces in which such functions are made uniform, furnish elegant solutions of some important problems in the theories of potential, electricity, light, sound, heat, and fluid motion. To give greater clearness to what follows, it may be well to take a simple illustration of a three-valued potential function in two dimensions, and show how to make it one-valued on a three-fold Riemann surface. Let (ρ, θ) be the polar coordinates of a

* Author's paper, loc. cit., p. 5, footnote.

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