NECESSARY CONDITIONS THAT THREE OR MORE PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER SHALL HAVE COMMON SOLUTIONS.

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IN a paper inspired by Hilbert's lectures in 1900, Yoshiye (*Mathematische Annalen*, volume 57) considers, among others, the following problem in the calculus of variations: To find the necessary conditions that the integral

$$\int_{u_0}^{u_1}[\lambda(z'-px'-qy')+\mu(p'-rx'-sy')+\nu(q'-sx'-ty')]\,du$$

shall vanish independently of the path of integration, whereby the two equations

$$F(x, y, z, p, q, r, s, t)=0, \quad G(x, y, z, p, q, r, s, t)=0$$

shall be satisfied. λ, μ, ν are arbitrary functions of u; p, q, r, s, t have the usual signification, i. e., $p=\partial z/\partial x$, $q=\partial z/\partial y$, etc.; the accents denote differentiation with respect to u.

The conditions which result upon consideration of this problem are that the two equations

$$\nu^2F_r+\mu\nu F_s+\mu^2F_t=0, \quad \nu^2G_r+\mu\nu G_s+\mu^2G_t=0$$

shall have a common solution in $\mu:\nu$, i. e., that the determinant

$$\begin{vmatrix} F_r & F_s & F_t & 0 \\ G_r & G_s & G_t & 0 \\ 0 & F_r & F_s & F_t \\ 0 & G_r & G_s & G_t \end{vmatrix}$$

shall vanish; and furthermore, that for the value of $\mu:\nu$ which satisfies these two equations the relation

$$\mu[F_t(G_y)-G_t(F_y)]+\nu[F_r(G_x)-G_r(F_x)]=0$$

shall be satisfied. F_x, F_y, etc., in the foregoing, denote partial derivatives with respect to x, y, etc.; the symbols (F_x), (F_y) are abbreviations for

$$F_x + F_z p + F_p r + F_q s \quad \text{and} \quad F_y + F_z q + F_p s + F_q t$$

respectively, and similarly for (G_x) and (G_y).

If one adds to the auxiliary conditions, in Yoshiye's problem, the third partial differential equation

$$H(x, y, z, p, q, r, s, t) = 0,$$

the first variation of the following integral must vanish:

$$\int_{u_0}^{u_1} [\lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty') + \xi F + \eta G + \zeta H]\, du$$

where ξ, η, ζ are Lagrange multipliers. The Lagrange equations for the determination of the functions $x, y, z, p, q, r, s, t, \lambda, \mu, \nu, \xi, \eta, \zeta$ are as follows:

1) $(\lambda p)' + (\mu r)' + (\nu s)' + \xi F_x + \eta G_x + \zeta H_x = 0,$

2) $(\lambda q)' + (\mu s)' + (\nu t)' + \xi F_y + \eta G_y + \zeta H_y = 0,$

3) $\lambda' + \xi F_z + \eta G_z + \zeta H_z = 0,$

4) $\mu' - \xi F_p - \eta G_p - \zeta H_p + \lambda x' = 0,$

5) $\nu' - \xi F_q - \eta G_q - \zeta H_q + \lambda y' = 0,$

6) $\mu x' - \xi F_r - \eta G_r - \zeta H_r = 0,$

7) $\mu y' + \nu x' - \xi F_s - \eta G_s - \zeta H_s = 0,$

8) $\nu y' - \xi F_t - \eta G_t - \zeta H_t = 0,$

9) $z' - px' - qy' = 0,$ 10) $p' - rx' - sy' = 0,$

11) $q' - sx' - ty' = 0,$

12) $F = 0,$ 13) $G = 0,$ 14) $H = 0,$

Eliminating λ', μ', ν' from 1) and 2) by means of 3), 4), and 5), we obtain

15) $\mu r' + \nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x) = 0,$

16) $\mu s' + \nu t' + \xi(F_y) + \eta(G\ ) + \zeta(H_y) = 0.$

Substituting in

$$\frac{dF}{du} = F_x x' + F_y y' + F_z z' + F_p p' + F_q q' + F_r r' + F_s s' + F_t t' = 0$$

for x', y', z', p', q', r', t' their values taken from 6), 8), 9), 10), 11), 15), 16), we obtain

$$\frac{\xi F_r + \eta G_r + \zeta H_r}{\mu}(F_x) + \frac{\xi F_t + \eta G_t + \zeta H_t}{\nu}(F_y)$$
$$- F_r \cdot \frac{\nu s' + \xi(F_x) + \eta(G_x) + \zeta(H_x)}{\mu} + F_s s'$$
$$- F_t \cdot \frac{\mu s' + \xi(F_y) + \eta(G\ ) + \zeta(H_y)}{\nu} = 0.$$

Two similar equations arise when we employ $dG/du = 0$ and $dH/du = 0$. Solving in these three equations for s', and using the further abbreviation

$$\mu[F_t(G_y) - G_t(F_y)] + \nu[F_r(G_x) - G_r(F_x)] \equiv (F, G),$$

whereby $(F, F) \equiv 0$ and $(F, G) = -(G, F)$, we obtain

$$\alpha) s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 G_t\} = (F, F)\xi + (G, F)\eta + (H, F)\zeta,$$
$$\beta) s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} = (F, G)\xi + (G, G)\eta + (H, G)\zeta,$$
$$\gamma) s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} = (F, H)\xi + (G, H)\eta + (H, H)\zeta.$$

From the last three equations we can deduce the following: Since $F = 0$ and $G = 0$ must have solutions in common, the left hand members of α), β) must have a common root in $\mu : \nu$; and for this common root (F, G) must vanish. It follows that for this same $\mu : \nu$ (H, F) and (H, G) also vanish, hence also that both members of γ) vanish. In other words, the left hand members of α), β), γ) have a common root in $\mu : \nu$, the condition for which is

$$\left\| \begin{matrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t \end{matrix} \right\| = 0.$$

For this common root in $\mu : \nu$ we have $(F, G) = 0, (G, H) = 0$ and $(H, F) = 0$, which yields

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{H_r(F_x) - F_r(H_x)}{H_t(F_y) - F_t(H_y)}.$$

If we inquire as to the possibility of common solutions to the four partial differential equations of second order

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0,$$

we obtain, by analogous procedure, the following four equations

$$\alpha') \quad s'\{\nu^2 F_r - \mu\nu F_s + \mu^2 F_t\} = (F, F)\xi + (G, F)\eta + (H, F)\zeta + (K, F)\theta,$$

$$\beta') \quad s'\{\nu^2 G_r - \mu\nu G_s + \mu^2 G_t\} = (F, G)\xi + (G, G)\eta + (H, G)\zeta + (K, G)\theta,$$

$$\gamma') \quad s'\{\nu^2 H_r - \mu\nu H_s + \mu^2 H_t\} = (F, H)\xi + (G, H)\eta + (H, H)\zeta + (K, H)\theta,$$

$$\delta') \quad s'\{\nu^2 K_r - \mu\nu K_s + \mu^2 K_t\} = (F, K)\xi + (G, K)\eta + (H, K)\zeta + (K, K)\theta.$$

Since $F = 0$, $G = 0$, and $H = 0$ have common solutions, the left hand members (and hence the right hand members) of $\alpha')$, $\beta')$, $\gamma')$ vanish for the same $\mu : \nu$; and for this $\mu : \nu$ we have

$$(F, G) = 0, \quad (G, H) = 0, \quad (H, F) = 0,$$

consequently also

$$(K, F) = 0, \quad (K, G) = 0, \quad (K, H) = 0;$$

and hence the left hand member of $\delta')$ vanishes for this same $\mu : \nu$. We obtain therefore, as a necessary condition that

$$F = 0, \quad G = 0, \quad H = 0, \quad K = 0$$

may have common solutions

$$\left\|\begin{matrix} F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_r & F_s & F_t \\ G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t & 0 \\ 0 & G_r & G_s & G_t & 0 & H_r & H_s & H_t & 0 & K_r & K_s & K_t \end{matrix}\right\| = 0.$$

And since

$$(F, G) = (F, H) = (F, K) = (G, H) = (G, K) = (HK) = 0$$

for the same value of $\mu : \nu$, we have, in addition to the vanishing of the above matrix,

$$\frac{F_r(G_x) - G_r(F_x)}{F_t(G_y) - G_t(F_y)} = \frac{F_r(H_x) - H_r(F_x)}{F_t(H_y) - H_t(F_y)} = \frac{F_r(K_x) - K_r(F_x)}{F_t(K_y) - K_t(F_y)}$$
$$= \frac{G_r(H_x) - H_r(G_x)}{G_t(H_y) - H_t(G_y)} = \frac{G_r(K_x) - K_r(G_x)}{G_t(K_y) - K_t(G_y)} = \frac{H_r(K_x) - K_r(H_x)}{H_t(K_y) - K_t(H_y)}.$$

Obviously the plan is general, and one could write down the necessary conditions that a system of n partial differential equations of the type above considered should have solutions in common.

NOTE ON DETERMINANTS WHOSE TERMS ARE CERTAIN INTEGRALS.

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THE object of the present note is to prove two simple identities involving a determinant whose elements are certain integrals, and to mention some special cases. Determinants of the form considered present themselves in problems connected with linear differential and integral equations and the calculus of