

of orthogonal functions, leads, if applied to the case $f = g$ to Bessel's identity

$$(6) \int_R \left[f - \sum_{p=1}^n \phi_p \int_R f \phi_p dR \right]^2 dR = \int_R f^2 dR - \sum_{p=1}^n \left[\int_R f \phi_p dR \right]^2,$$

from which Bessel's inequality immediately follows.

5. Theorems analogous to those of the present note involving finite sums or infinite series in place of integrals may be proved in a similar manner.

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ON THE TACTICAL PROBLEM OF STEINER.

BY PROFESSOR W. H. BUSSEY.

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THE study of tactical configurations known as triple systems had its origin in two problems proposed independently by J. Steiner* and T. P. Kirkman.† The Steiner problem, which is the more general and includes the other, is as follows:

For what values of n is it possible to arrange n elements in sets of three, called triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements that cannot be obtained from it by a mere permutation of the elements? When such an arrangement in triads has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three that is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four that is not a tetrad and does not contain a triad is contained in one and only one pentad? In general, when an arrangement in k -ads has been made, is it possible to arrange the n elements in sets of $k+1$, called $(k+1)$ -ads so that no l -ad ($l \leq k$) is contained in a $(k+1)$ -ad, and so that every set of k elements that is not a

* *Journal für die reine und angewandte Mathematik*, vol. 45, p. 181.

† *The Lady's and Gentleman's Diary* for 1850. For other references to the literature of Kirkman's fifteen school girls problem see Ball's *Mathematical Recreations and Essays*, 4th edition, page 121.

k -ad and does not contain an l -ad ($l < k$) is contained in one and only one $(k + 1)$ -ad?

The part of the problem that relates to triads has been completely solved.* The other parts have been little studied.

If an arrangement of n elements in triads, tetrads, pentads, etc., is possible, the number of k -ads for $k = 3, 4, 5, \dots$ is given by the formula

$$N_k = \frac{1}{k!} n(n-1)(n-3) \cdots (n - [2^{k-2} - 1]).$$

This formula was suggested by Steiner. It may be proved without much difficulty by complete induction.

This paper has to do with the case in which n is a number of the form $2^j - 1$. Its object is to show that it is possible to arrange such a number of elements in k -ads for $k = 3, 4, 5, \dots, j + 1$. The formula gives $N_k = 0$ when $k > j + 1$.

Consider the $2^{k+1} - 1$ elements $(x_1, x_2, x_3, \dots, x_{k+1})$, each x being 0 or 1 and the element $(0, 0, 0, \dots, 0)$ being excluded. For convenience the language of geometry is used and each of the elements is called a point. The $2^{k+1} - 1$ points are said to constitute a finite geometry of k dimensions, or, more briefly, a k -space.† Consider also the linear homogeneous congruence, modulo 2,

$$(1) \quad a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{k+1} x_{k+1} \equiv 0,$$

in which each coefficient is 0 or 1 and at least one of them is not zero. The points of the k -space that satisfy such a congruence are said to constitute a $(k - 1)$ -space; the points that satisfy two linearly independent congruences of the type (1) are said to constitute a $(k - 2)$ -space; and, in general, the points that satisfy $(k - l)$ linearly independent congruences of type (1) are said to constitute an l -space. The number of solutions of a set of congruences of type (1) may be counted without much difficulty and the number of points in an l -space, $l < k$, found to be $2^{l+1} - 1$. In particular, the number of points in a plane (2-space) is seven, and the number in a line (1-space) is three. A single point constitutes a 0-space. The points common to two l -spaces, if there are any, constitute an r -space, where $0 \leq r \leq l - 1$. A set of $l + 1$ points which are

* *Encyclopédie des Sciences mathématiques*, vol. 1, p. 80.

† See Veblen and Bussey, "Finite projective geometries," *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 241-259. In particular, see § 2.

not all contained in the same $(l - 1)$ -space is contained in one and only one l -space. The $l + 1$ points of such a set, if taken l at a time, determine a number of $(l - 1)$ -spaces whose points constitute a set that may conveniently be called a *simplex** of order l . The $l + 1$ points are called vertices. A convenient symbol for a simplex of order l is $S(l + 1)$. Any $i + 1$ of the vertices of a $S(l + 1)$ are the vertices of a simplex $S(i + 1)$ whose points are all contained in the $S(l + 1)$.

THEOREM. *The number of points in a simplex of order l is one less than the number of points in the l -space determined by its $l + 1$ vertices.*

By actual count, the theorem is true for $l \leq 3$. The rest of the proof consists in showing that it can be proved for a simplex $S(m + 1)$ if it be assumed true for every simplex $S(l + 1)$ for which $l < m$. This is done by arranging the points of the simplex $S(m + 1)$ in the m following sets. The sets are not mutually exclusive.

1. The $m + 1$ vertices of the simplex $S(m + 1)$.
2. The points of the ${}_{m+1}C_2$ lines determined by the vertices taken two at a time.
3. The points of the ${}_{m+1}C_3$ planes determined by the vertices taken three at a time.
4. The points of the ${}_{m+1}C_4$ 3-spaces determined by the vertices taken four at a time.

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$i + 1$. The points of the ${}_{m+1}C_{i+1}$ i -spaces determined by the vertices taken $i + 1$ at a time.

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n . The points of the ${}_{m+1}C_m$ $(m - 1)$ -spaces determined by the vertices taken m at a time.

[Note: The symbol ${}_{m+1}C_j$ means the number of combinations of $m + 1$ things taken j at a time.]

The set numbered $i + 1$, i being any one of the numbers 1, 2, 3, . . . , m , consists of the points contained in ${}_{m+1}C_{i+1}$ i -spaces each of which is determined by $i + 1$ of the vertices of the simplex $S(m + 1)$ or, in other words, by the $i + 1$ vertices of a simplex $S(i + 1)$ which is contained in the simplex $S(m + 1)$. By hypothesis, each of these i -spaces contains one and only one

*The word is used in geometry of n -dimensions to denote the configuration analogous to the triangle in the plane or the tetrahedron in 3-space.

point not contained in the simplex $S(i + 1)$ that determines it. But that one point is a point of the simplex $S(m + 1)$ by the very definition of simplex. Therefore, if one begins to count with the first set and counts through the sets in order, the number of points in the set numbered $i + 1$ that have not been counted in any previous set is ${}_{m+1}C_{i+1}$. It follows that the number of points in the simplex $S(m + 1)$ is

$$\sum_{j=1}^m {}_{m+1}C_j = 2^{m+1} - 2,$$

which is one less than the number of points in the m -space determined by the $m + 1$ vertices of the simplex.

From this theorem it follows that the $l + 1$ vertices of a simplex of order l determine uniquely another point, namely, the one point of the l -space determined by the simplex that is not also a point of the simplex. It is convenient to call this point the point complementary to the simplex. The triads, tetrads, pentads, etc. of the Steiner problem are found as follows: Every simplex $S(2)$ determines a triad consisting of its two vertices and the complementary point; every simplex $S(3)$ determines a tetrad consisting of its three vertices and the complementary point; and, in general, every simplex $S(l - 1)$, $l \leq k + 2$, determines an l -ad consisting of the $l - 1$ vertices and the complementary point. There are no l -ads for $l > k + 2$.

When $n = 2^6 - 1 = 63$, it is possible to arrange the n elements in triads, tetrads, pentads, hexads, and heptads. There is no arrangement of the 63 elements in l -ads for $l > 7$. This special case was involved in Steiner's investigation of the configuration of the 28 double tangents of a quartic curve* and led him to propose for solution the "Combinatorische Aufgabe" which I have called "The tactical problem of Steiner."

ON THE SO-CALLED GYROSTATIC EFFECT.

BY PROFESSOR ALEXANDER S. CHESSIN.

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IN computing the resisting couple of gyrostats or the so-called "gyrostatic effect" it is customary to assume that it is equal to $C\lambda\omega \sin \theta$, where C , λ , ω and θ denote respectively the moment of inertia of the gyrostat about its geometrical axis, the angular

* *Journal für die reine und angewandte Mathematik*, vol. 49, pp. 265-272.