SHORTER NOTICES.

Vorlesungen über die Elemente der Differential- und Integralrechnungen und ihre Anwendungen zur Beschreibung von Naturerscheinungen. Von HEINRICH BURKHARDT. Leipzig, Teubner, 1907. 8vo. xi + 252 pp.

THIS volume has been prepared to meet the needs of a growing body of students who are finding the calculus useful in certain sciences which do not call for its fuller development— notably chemistry, mineralogy, and statistics. The author has aimed so to choose the material for exposition that the volume may serve as a first course in the calculus for others who, from choice or necessity, require an advanced course. The problem is complicated by seeking to avoid all "arithmetization" of material such as would be necessary for a complete and rigorous treatment of the fundamental limit

$$\lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^n.$$

This limit is, in a sense, the pons asinorum of the calculus and Professor Burkhardt feels that the chemist can not be gotten over it easily, and that even the mathematician finds difficulty with it when he is introduced to it too abruptly and not in connection with a complete discussion of the concept of limit. (Preface, page v.) The avoidance of arithmetic necessitates, from the author's point of view, leaving out all reference to convergence and to infinite series. Thus the usual series are spoken of as formulas and treated as approximations to the value of a function for certain of its arguments.

It is certainly interesting and instructive to any teacher of mathematics to know how a mathematician of Professor Burkhardt's attainments seeks to solve the problem he has set for himself.

The book is divided into ten chapters and a Nachtrag. It has a good table of contents and a complete register and is put together in the well known style of the publishers. It begins by a long, but in no wise uninteresting, introduction leading up, by easy stages through the laws of motion, to the notion of differential quotient. In this first chapter one finds the usual concepts and symbols together with an exposé of the fundamental principles of cartesian geometry as applied to the straight line and to tangents in general.

The second chapter is devoted to the differentiation of rational functions, one section being set aside for the special consideration of the linear fractional function. This enables the author to illustrate the notion of infinitely great and to introduce the hyperbola and its asymptotes.

The differentiation of irrational functions is considered in the third chapter. The rule for differentiating inverse functions is first taken up and, by its aid, the rule for differentiating x^m is shown to hold for any rational m. The chapter closes with the rule for differentiating a function of a function.

The student is now able to differentiate any algebraic function, and in the next chapter he is introduced to the inverse operation. The chapter opens by finding a function whose differential quotient is gt, without previous knowledge of the differential calculus. This is the nearest approach to the treatment of a definite integral as the limit of a sum. Such a treatment would, of course, be impossible from the author's point of view. Integration is defined as the inverse of differentiation, and the integral of x^m is found for all values of m except $m = -1$. We find here the geometrical interpretation of the definite integral as the area beneath the curve whose ordinates, between the limits, are given by the integrand. The trapezoidal formula for calculating approximately the value of a definite integral is followed by its application to the integral

$$\int_1^2 \frac{dx}{x}.$$

Chapter five contains the crux of the whole volume. Professor Burkhardt takes his dilemma by both horns, so to speak, and *defines*

$$\int_1^x \frac{d\xi}{\xi}$$

to be the natural logarithm of x. By means of the geometrical interpretation just given, the student is shown that this new function is defined for all positive values of x and that it is not defined for $x = 0$. The question of its definition for negative values of x is expressly excluded.

The addition theorem for logarithms follows easily from the definition, and a comparison with common or Brigg's logarithms leads to the formula

$$\log x = \frac{\log z}{\log \text{Brigg } z} \cdot \log \text{Brigg } x.$$

This formula enables one to calculate the natural logarithms of all positive numbers, with the help of a table of common logarithms, as soon as the natural logarithm of *any* number $z \neq 1$ is known. The natural logarithm of 2 was found approximately in the preceding chapter. Thus we are across the bridge and the student has not realized its existence, the approaches have been so cunningly concealed !

The reviewer is reminded of the proof for differentiating the logarithm given in Olney's Calculus (1871) and there credited to the astronomer, J. C. Watson. Olney says of this proof that it "banishes from the calculus the last necessity for resort to series to establish any of its fundamental operations." But the modulus of the common system — there's the rub ! Olney resorts to infinite series to find it and Burkhardt relies upon the trapezoidal formula, which is dangerously near to the same thing. The fact is, it seems a doubtful policy to avoid all reference to arithmetic and substitute therefor the notion of an integral as a function of its upper limit. The student must feel that the method is, to say the least, round about. The question is pedagogically difficult, but teachers will find greater satisfaction in treating it as is done, for example, in Osgood's Calculus even for the student of chemistry or statistics.

The remainder of the chapter is devoted to the integration of rational fractional functions but only for the cases where the denominator has distinct or multiple real roots. Application is made to the integration of simple ordinary differential equations expressing chemical reaction between two or more substances.

The sixth chapter is the longest in the book and in many respects the most interesting. It contains sections upon higher derivatives, the mean value theorem, Maclaurin's and Taylor's formulas. The three important special cases, the binomial formula, the exponential formula, and the logarithmic or Mercator formula, are emphasized and illustrated. Numerical calculation is given a prominent place, and sections on reckoning with small quantities, on approximate solution of numerical equations, and reversion of approximate formulas are given. Section 50, on the generalized mean value theorem and the remainder in the Maclaurin formula, is the nearest approach to a complete treatment of the development of a function in series which Professor Burkhardt allows himself. Maxima and minima values of a function receive only the barest treatment and in only the simplest cases. The same thing is true of the evaluation of indeterminate forms. One may, of course, disagree with Professor Burkhardt's placing of emphasis upon the materials chosen for this chapter. It would seem that maxima and minima values of a function were of sufficient importance, even to the chemist, to deserve fuller notice. But one must agree that the chapter is charmingly written and will commend itself to teachers for its simple straightforward presentation of subjects which many students find tedious and difficult in their formal setting.

Interpolation forms the subject of the seventh chapter. It is first performed by setting up a rational integral function to fit the observations, and a problem is fully worked out with all the necessary numerical calculation of coefficients. In the next place it is performed by finite differences, and lastly where the number of observations is greater than the number of constants to be determined.

Some of the numerical calculation might well have been left to the student and a paragraph or two devoted to logarithmic plotting, since it is just as easy to fit the curve $y = bx^a$ to the observations by plotting the logarithms of x and y and then using the straight edge as it is to fit $y = ax + b$ to the observations in the first place (page 184), and it may turn out to fit more accurately.

Chapters eight and nine are each but twelve pages in length. The first treats of the infinitesimal or differential, and the second of functions of two variables and partial derivatives.

In the tenth chapter, the student is led to see the use of trigonometric functions in describing periodic phenomena. Rules for differentiating and integrating trigonometric functions follow, and by inversion for the circular or *cyclometric* functions. Trigonometric interpolation is considered, as well as three simple cases of vibration, viz., free vibration, damped vibration, and forced vibration, in case the external force is itself periodic.

The Nachtrag is devoted to interpolation by means of exponential functions.

The book, as a whole, commends itself for its simplicity of presentation. The treatment of the logarithm is a doubtful pedagogical expedient but there is no lax rigor about it. The responsibility is shifted to the numerical calculation of logarithms, just as was done in Olney's Calculus over thirty years ago.* A student who has had elementary training in algebra and trigonometry can read the book without difficulty and, in the main, it presents enough of the calculus and its applications to serve that body of students of which we have spoken at the beginning. But it does not represent what we, in America, have come to consider as a first course in the calculus.

L. WAYLAND DOWLING.

Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen. Von VICTOR VON DANTSCHER. Leipzig und Berlin, Teubner, 1908. vi + 79 pp.

IN the preface we are told that these lectures are based upon a course given by Weierstrass in the summer semester of 1872, which was followed by the author of the present volume, and upon an elaboration of a later course given in 1884. The work under review is, however, not a mere reproduction of things given by Weierstrass, but it is the direct outcome of a course given repeatedly at the University of Gratz by Professor von Dantscher. It furnishes an easy and clear introduction to that theory of irrational numbers which was first developed by Weierstrass in his lectures at the University of Berlin, and it has decided pedagogic as well as scientific value.

C. Méray was the first to give a purely arithmetic meaning to the term irrational number,† and the theories developed by him, G. Cantor, Heine, and Dedekind have perhaps become better known than the theory of Weierstrass. This may be partly due to the fact that no expository publication relating to this theory was ever prepared under the direction of Weierstrass, and only the fundamental elements of this theory have been accessible in the works of Kossak, Pincherle, Biermann, and others. The first of these was based upon a course of lectures given by Weierstrass during the winter semester of 1865–6, and it was published in 1872 under the title "Die Elemente

* It should be stated that Olney sought only to avoid infinite series. The Watson proof of the rule for differentiating the logarithm tacitly assumes that $dx^n/dx = nx^{n-1}$ holds for all real values of n.

† Encyclopédie des Sciences mathématiques, tome I, vol. 1, p. 149.