INTEGRAL EQUATIONS.

An Introduction to the Study of Integral Equations. By MAXIME BÔCHER. Cambridge Tracts in Mathematics and Mathematical Physics, No. 10. Cambridge, The University Press, 1909. iv + 72 pages.

IN 1823 Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind, and in 1837 Liouville showed that the determination of a particular solution of a linear differential equation of the second order could be effected by solving an integral equation of a different type, called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased at first but slowly. Recently, however, stimulated by the researches of Volterra, Fredholm, and Hilbert in the period between 1896 and the present time, that which seemed at first only a ripple has grown into a formidable wave which bids fair to carry the integral equation theory into a place beside the most important of the mathematical disciplines. Notwithstanding the rapidly multiplying investigations in integral equations and the numerous applications of them which have been made, the sources of information concerning the theory have remained widely scattered and none too easily accessible to any but the specialist in the subject. It is with a hearty welcome, therefore, that the thoughtful mathematician will receive an introduction to the theory written by so clear a thinker and writer as the author of the book which is the subject of this review. As stated in the preface, the purpose of the author was to furnish the careful student with a firm foundation for further study, and at the same time so to display and arrange the principal theorems that one may with only a superficial reading obtain some idea of the subject. These objects seem to have been successfully attained. The book should furthermore be very useful as a text in an introductory course, especially if the instructor would content himself at first with the discussion of integral equations whose kernels are continuous or have discontinuities of the explicit forms which occur in the problem of Abel, treated in § 2, and other applications.

The emphasis which is placed on the historical development of the subject is an interesting feature of the book. After an introductory section in which some essential theorems concerning definite integrals are set down, the problems of Abel and Liouville, probably the earliest applications of integral equations, are discussed, and thereafter the reader's attention is constantly directed to the contributions which have been made by Volterra, Fredholm, Hilbert, Schmidt, Kneser, and other writers. In the papers of Hilbert and Schmidt the kernel of the integral equation is first assumed to be symmetric and continuous. Later they show that the theory for unsymmetric kernels can be regarded as an application of the theory for the symmetric case. Professor Bôcher, following the earlier writers, has inverted this arrangement, which seems more convenient since many of the principal results follow as easily for the unsymmetric as for the symmetric hypothesis. The author has also admitted from the start certain kinds of discontinuities in the kernels of his equations. This is perhaps disconcerting to the reader who wishes merely a survey of the theory, but the applications of equations with discontinuities are so frequent that one must feel that the admission is justified. It seems regrettable that more of the applications of the integral equation theory, for example Hilbert's unification of the theories of the expansion of an arbitrary function in terms of other functions and some of the applications to boundary value problems, could not have been introduced. The limited size of the book was evidently the preventive.

In commenting upon the theory as developed by Professor Bôcher, I shall not attempt to follow closely the order of his arrangement, but shall try to give an idea of the contents of the book as they have impressed themselves upon me. There are two kinds of integral equations which may be written in the forms

$$f(x) = \int_a^b K(x, \xi)\, u(\xi) d\xi, \tag{1}$$

$$u(x) = f(x) + \int_a^b K(x, \xi)\, u(\xi) d\xi, \tag{2}$$

where $K(x, \xi)$ and $f(x)$ are given functions, while $u(x)$ is to be determined. The problem proposed by Abel was to determine a curve $y = y(x)$ down which a heavy particle would fall from a variable point (x, y) to the origin $(0, 0)$ in a time $T = f(x)/2g$, where $f(x)$ is an arbitrarily assigned function. The integral equation which gives the solution is one of the first kind, in which $K(x, y) \equiv 0$ for $y > x$. It has the form

$$f(x) = \int_0^x \frac{v'(\xi)d\xi}{\sqrt{(x-\xi)}},$$

where $v(x)$ is the length of arc measured from the origin and is to be determined. Liouville later made the determination of a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + [\rho^2 - \sigma(x)]y = 0,$$

where ρ is a constant, depend upon the solution of the integral equation

$$(3) \qquad u(x) = \cos \rho (x - a) + \frac{1}{\rho}\int_a^x \sigma(\xi) \sin \rho(x - \xi)u(\xi)d\xi,$$

which is an equation of the second kind. In §§ 3, 5 Professor Bôcher exhibits the method which Liouville applied to the solution of these two equations, and applies the same method of successive substitutions to the general equation of the second kind.

The treatment devised by Volterra for equations of the second kind is both remarkable and elegant. It depends upon the notion of the iterated functions $K_i(x, y)$ defined by the formulas

$$K_1(x, y) = K(x, y), \quad K_i(x, y) = \int_a^b K(x, \xi)K_{i-1}(\xi, y)d\xi.$$

The series

$$(4) \qquad - k(x, y) = K_1 + K_2 + \cdots,$$

when it is uniformly convergent, determines uniquely a continuous function $k(x, y)$ which with $K(x, y)$ satisfies the equations

$$(5) \qquad K(x, y) + k(x, y) = \int_a^b K(x, \xi)k(\xi, y)d\xi$$

$$= \int^b k(x, \xi)K(\xi, y)d\xi.$$

Any two functions K, k which have proper continuity properties and satisfy the last equations are said to be "reciprocal." By means of equations (5) it can be shown that the integral equation of the second kind has one and only one continuous solution, which is expressed by the formula

$$(6) \qquad u(x) = f(x) - \int_a^b k(x, \xi) f(\xi) d\xi.$$

The solution by Volterra which has just been discussed depends for its validity upon the convergence of the series (4). Another method suggested by Volterra, but investigated by Fredholm, and later revised and extended by Hilbert, goes deeper into the meaning of the integral equations, explains the circumstances under which the reciprocal function $k(x, y)$ will or will not surely exist, and has besides an important application to integral equations involving an arbitrary parameter λ which will be mentioned later. Professor Bôcher shows in §7, following Fredholm, how one may regard the equation (2) as a limiting case for the system of equations

$$(7) \qquad u_n(x_i) = f(x_i) + \sum_{j=1}^{n} K(x_i, x_j) u_n(x_j) \qquad (i = 1, 2, \cdots, n)$$

as n becomes infinite. Here $x_1, x_2, \cdots, x_n = b$ are supposed to divide the interval ab into n equal parts, and $u_n(x_1), u_n(x_2), \cdots, u(x_n)$ are the quantities to be determined. The determinant D_n of these equations goes over as n approaches infinity into an infinite series of integrals involving the kernel K, and if the values (x_μ, x_ν) have the limit (x, y) the corresponding cofactor $D_n(x_\mu, x_\nu)$ of D_n approaches a limiting value $D(x, y)$ which is also expansible into an infinite series. In § 8 the convergence of the series for D and the "adjoint" $D(x, y)$ is rigorously proved, and the important relations

$$-DK(x, y) + D(x, y) = \int_a^b K(x, \xi) D(\xi, y) d\xi$$
$$= \int_a^b D(x, \xi) K(\xi, y) d\xi$$

are derived. It follows at once that when $D \neq 0$ the function

$$(8 \qquad k(x, y) = -\frac{D(x, y)}{D}$$

is reciprocal to $K(x, y)$, and equation (2) has a unique solution expressed by means of formula (6). On the other hand when $D = 0$ there will be no solution unless $f(x)$ satisfies

$$\int_a^b D(x, \xi)f(\xi)d\xi = 0,$$

a relation which is suggested by the condition which must be satisfied if equations (7) have a solution when $D_n = 0$.

The integral equation (3) which Liouville studied is of the form

$$u(x) = f(x) + \lambda \int_a^b K(x, \xi)u(\xi)d\xi, \tag{9}$$

where $\lambda = 1/\rho$ and $K(x, y) \equiv 0$ for $y > x$. In § 9 Professor Bôcher begins the study of such equations. The determinant and adjoint function, as well as the reciprocal function, are here functions of the form $D(\lambda)$, $D(x, y, \lambda)$, $k(x, y, \lambda)$ containing the parameter λ which may take either real or complex values. The roots of $D(\lambda)$ are the "Eigenwerte" of Hilbert, or the "roots for the function $K(x, y)$." It is found in §§ 9, 10 that the necessary and sufficient condition that $K(x, y)$ have a reciprocal $k(x, y, \lambda)$ corresponding to a particular value of λ is that $D(\lambda) \neq 0$. If this condition is satisfied, equation (9) has a unique solution determined by equations (8) and (6).

The situation is somewhat different for the homogeneous equation

$$u(x) = \lambda \int_a^b K(x, \xi)u(\xi)d\xi, \tag{10}$$

as is explained in § 10. The unique solution of this equation when $D(\lambda) \neq 0$ is $u = 0$. On the other hand, for any root of $D(\lambda)$ the homogeneous equation has always an infinity of continuous solutions, called "principal solutions," which do not vanish identically. When $D = 0$ it follows from these results that the non-homogeneous equation (9) has either no continuous solution or else an infinite number found by adding to any particular solution of (9) the solutions of (10).

It was mentioned above that in the papers of Hilbert and Schmidt the theory of integral equations with unsymmetric kernel K has been made to depend upon that of equations in which the kernel is symmetric. In §§ 11, 12 Professor Bôcher develops the theorems which relate especially to equations with symmetric kernels. For any such equation the determinant $D(\lambda)$ has at least one root, all the roots are necessarily real, and to any root of $D(\lambda)$ there corresponds only a finite number of linearly independent principal solutions of the homogeneous equation (10). A system $u_i(x)$ $(i = 1, 2, \cdots)$ of principal solutions belonging to roots of $D(\lambda)$ can be so chosen that any principal solution of equation (10) is expressible linearly and with constant coefficients in terms of a finite number of the functions $u_i(x)$, and furthermore so that

$$\int_a^b u_i^2(x)dx = 1, \quad \int_a^b u_i(x)u_j(x)dx = 0 \qquad (i \neq j).$$

A system of solutions having these properties is called "a complete normalized orthogonal system of characteristic functions for the kernel K." The trigonometric functions $\sin x$, $\sin 2x$, $\cdots$, are an example of such a system, in terms of which any function satisfying suitable restrictions can be expanded as an infinite series. Similar expansion theorems hold also for the system of characteristic functions belonging to any symmetric kernel. Professor Bôcher has restricted himself here, however, to the consideration of a single expansion, that for the kernel $K(x, y)$ itself, and to some of its applications.

The theory of the integral equations of the second kind having been developed, it is a comparatively simple matter to show, as Professor Bôcher does in § 13, that the solution of the integral equation

$$(11) \qquad f(x) = \int_a^x K(x, \xi)u(\xi)d\xi,$$

which is one of the first kind with $K \equiv 0$ when $y > x$, are all solutions of the equation

$$f'(x) = K(x, x)\,u(x) + \int_a^x \frac{\partial K(x, \xi)}{\partial x}\,u(\xi)d\xi.$$

If $K(x, x)$ does not vanish in the interval ab, this is an equation of the second kind (2), and the problem of solving it is equivalent to the solution of the original equation (11) of the first kind. The case when $K(x, x)$ vanishes identically is treated, and an example illustrative of the case when $K(x, x)$ has a finite number of zeros is given. The section concludes with the study of the more general equation of the first kind where $K(x, y)$ is assumed to have a discontinuity along a curve $y = \phi(x)$.

In the section just described the kernel was assumed to be finite. The earliest integral equation of the first kind, that of Abel, was however one in which the kernel became infinite along the line $x = y$. The concluding section of the book is devoted to equations of the type

$$f(x) = \int_a^x \frac{G(x, \xi)}{(x-\xi)^\lambda} u(\xi) d\xi,$$

which has a kernel with an infinite discontinuity including Abel's kernel as a special case for $G = 1$, and to a number of examples not falling under the previous theory. Especially interesting is the explanation of the relation of Fourier's integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos(x\xi) \cos(\xi\xi_1) f(\xi_1) d\xi_1 d\xi$$

to the theory of integral equations in which the limits are n-finite.

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SHORTER NOTICES.

Grundlagen der Analysis. Von MORITZ PASCH. Ausgearbeitet unter Mitwirkung von CLEMENS THAER. Leipzig, Teubner, 1908. 8vo. vi + 140 pp.

THIS book presents an admirable attempt to develop the concept of the real number in a more exact logical fashion. There is no attempt to reduce the assumptions to a categorical set, and even their consistency is not considered; but they are everywhere clearly stated, the theorems follow by ready deductions, and the large number of definitions would seem to be put in an unusually clear way, and one especially well adapted to the purpose of the general argument.

The book opens with a consideration of the relation of things to names, of the notions of precede and follow, and of methods of mathematical proofs. This is followed by a treatment of sets, sequences, and series, leading up to integers. By subjecting the integers to the four fundamental operations, fractions, including decimal fractions, and negative numbers are intro-