THE THIRD REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE third regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Missouri, on Saturday, November 27, 1909. The following members of the Society were present:

Professor L. D. Ames, Professor W. C. Brenke, Professor E. W. Davis, Professor G. R. Dean, Professor L. M. Defoe, Mr. A. B. Frizell, Mr. E. S. Haynes, Professor E. R. Hedrick, Dr. Louis Ingold, Professor O. D. Kellogg, Professor W. H. Roever, Dr. Mary S. Walker, Dr. Paul Wernicke, and Professor W. D. A. Westfall.

The morning session was opened at 10 A.M. and the afternoon session at 2 P.M., Professor Davis presiding. Lincoln, Nebraska, was decided upon for the next meeting, and the following program committee was elected: Professor Davis (chairman), Dr. Wernicke, Professor Kellogg (secretary).

The following papers were presented:

(1) Professor G. O. JAMES: "On the reduction of time observations in vertical circle through Polaris."

(2) Professor L. D. AMES: "Some theorems of Lie on ordinary differential equations."

(3) Professor J. B. SHAW: "Scalars of lineal products."

(4) Professor M. B. PORTER: "On Fourier sequences."

(5) Dr. PAUL WERNICKE: "Note on sine theorems in hyperspace."

(6) Professor W. H. ROEVER: "The southerly deviation of falling bodies."

(7) Professor G. R. DEAN: "Stresses in an isotropic plate."

(8) Professor E. W. DAVIS: "A paradox relating to the imaginary line."

(9) Professor W. D. A. WESTFALL: "The continuity of integral rational functions of infinitely many variables."

(10) Professor H. B. NEWSON: "On linear groups in two variables."

(11) Professor W. C. BRENKE: "Summation of a series of Bessel's functions by means of an integral."

(12) Professor E. R. HEDRICK: "On a property of assemblages whose derivatives are closed."

(13) Mr. A. B. FRIZELL: "Natural numbers defined by the principles of abstract groups."

(14) Dr. LOUIS INGOLD: "Outline of a vector theory of curves."

In the absence of the authors, the papers of Professors James, Shaw, and Newson were read by title. Abstracts of the papers follow below.

1. From the rigorous formulas for the reduction of time observations in the vertical circle through Polaris Professor James develops the chronometer correction in a power series in p_0, the polar distance of Polaris, neglecting terms of the third and higher powers in p_0. By the introduction of the reduction to the pole, published each year in the *American Ephemeris and Nautical Almanac*, this series is transformed into one in which the term in p_0^2 is missing, thus reducing the approximate formula to a single term. The error of this simple formula is investigated and latitudes for which it is admissible determined. An analogous development and transformation is made for azimuth, and the resulting reduction formulas are in both cases adapted to the engineer's transit.

2. Consider an ordinary differential equation $Xy' - Y = 0$ from the standpoint of the Lie theory. Consider the three following statements: (A) Its integral curves admit the group

$$Uf = \xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y}.$$

(B) The equation itself admits the extended group. (C) An integrating factor is $1/(X\eta - Y\xi)$. Professor Ames's paper states three well-known theorems: (1) that (B) and (C) are equivalent, (2) that (A) and (B) are equivalent, (3) that (A) and (C) are equivalent. Theorem (1) was proved in a previous paper. The other two are proved in the present paper. The statements and proofs are given in analytic terms without making any use of the group theory, and the three theorems are proved independently of one another. Of course any one follows from the other two.

3. A lineal product as defined by Grassmann is one whose properties are independent of the ground or system of units in

terms of which the numbers are expressed. Scalars of such products would have the same property. Thus for quaternions $S\cdot\alpha\beta\gamma = -S\cdot\beta\alpha\gamma$, whatever numbers α, β, γ may be. Scalars of lineal products must therefore be expressible in terms of more elementary scalars of lineal products, or else be themselves elementary scalars.

Professor Shaw's paper discusses the scalar defined by the recurrence formula $I\cdot\alpha_1\alpha_2\cdots\alpha_{2m} = I\cdot\alpha_1\alpha_2 I\cdot\alpha_3\cdots\alpha_{2m} + I\cdot\alpha_1\alpha_3 I\cdot\alpha_4\cdots\alpha_{2m}\alpha_2 + I\cdot\alpha_1\alpha_4 I.\alpha_5\cdots\alpha_{2m}\alpha_2\alpha_3 + \cdots + I\cdot\alpha_1\alpha_{2m} I\cdot\alpha_2\cdots\alpha_{2m-1}$. This scalar is a Pfaffian and is of much use. Scalars of corresponding types of the forms $I\cdot\alpha_1\cdots\alpha_{3m} = \Sigma\cdot I\cdot\alpha_1\alpha_2\alpha_3\cdot I\cdot\alpha_4\cdots\alpha_{3m}$, etc., are slightly examined.

4. If

$$\sum_1^n (a_n \sin nx + b_n \cos nx) = S_n$$

be the Fourier sequence of order n of the function $f(x)$ when it is supposed that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx$$

exists, Professor Porter shows that from any sequence S_{n_i} $(i = 1, 2, \cdots)$ a sequence can be picked out which converges over a point set of measure 2π to the value $f(x)$.

5. The formula used for the volume belonging to the n-dimensional "simplex" on page 212 of the February, 1909, BULLETIN $V = p_{01}p_{12} \sin (P_{01}, P_{12})\cdot p_{23} \sin (P_{12}, P_{23})\cdot \sin (P_{012}, P_{123})\cdots$ is capable of extension. It remains correct

1) if we replace the p_{ij} by the n joins p_{0j} of one point P_0 to the remaining n; and simultaneously $P_{ijk}, \cdots$ by P_{0jk};

2) if, after permuting the p_{ij} (and P_{ij}), we replace $P_{0123\cdots\nu}$ by any ν-plane parallel to the first, second, $\cdots$, νth P_{ij} (in the new order);

3) if we permute the vertices, hence the indices $0, 1, \cdots, n$, throughout.

Equating expressions thus recognized to be equal, Dr. Wernicke obtains theorems connecting the sines. For example, in 3-space,

$$p_{12} \sin (P_{01}, P_{12}) \sin (P_{012}, P_{123}) = p_{13} \sin (P_{01}, P_{13}) \sin (P_{013}, P_{132})$$

which, when $p_{12} = p_{13}$, becomes the sine theorem of spherical trigonometry.

As a further example, in 4-space, the angles made by the 2-planes P_{012} and P_{034} are connected with those made by P_{014} and P_{023} by the sine theorem

$$\sin(P_{01}, P_{02}) \sin(P_{03}, P_{04}) \sin_1(P_{012}, P_{034}) \sin_2(P_{012}, P_{034}) = \sin(P_{01}, P_{04}) \sin(P_{02}, P_{03}) \sin_1(P_{013}, P_{024}) \sin_2(P_{013}, P_{024}),$$

$\sin_1$, $\sin_2$ denoting the sines of their "first" and "second" angles, respectively. The paper will be offered to the *Annals of Mathematics* for publication.

6. A body dropped from a point P_0, attached to the rotating earth, falls in a certain curve c, in a field of force F_1, which is fixed in space. It receives an initial velocity which is the same as the velocity of P_0 at the instant it is dropped. A plumb-line supported at P_0 hangs in a field of force F_2 which rotates with the earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where r and z denote the distances of a general point from the axis of rotation OZ and the plane of the equator of the earth) are force functions of the fields F_1 and F_2 respectively,

$$U_1 + \tfrac{1}{2}\omega^2 r^2 = U_2,$$

in which ω denotes the angular velocity of the earth's rotation.

Professor Roever reduces the problem to one in two dimensions by the following device: The curve c, when rotated about the earth's axis, generates a surface of revolution of axis OZ. The meridian curve of this surface which lies in the plane of OZ and P_0 we call curve (1). Plumb-lines of different lengths supported at P_0 do not coincide. The locus of the plumb-bobs of all plumb-lines supported at P_0, which lies in the plane of OZ and P_0, we call curve (2). Curves (1) and (2) are tangent at P_0 to the line of force of the field F_2 which passes through P_0. In order to establish the existence of a southerly deviation (in the northern hemisphere) it is enough to show that curve (1) lies south of curve (2). The magnitude of the deviation is the distance between the points in which these curves pierce the equipotential surface $U_2 = K$ which represents the earth's surface.

In particular, the form of the function U_1 was assumed to be M/ρ, where M is the mass of the earth and ρ is the distance of a general point from the earth's center O. For this function the equipotential surface $K = U_2 = M/\rho + \frac{1}{2}\omega^2 r^2$, which is taken to represent the earth's surface, is approximately a spheroid of ellipticity .0017. (That of the earth is about .0034). The common tangent and the common normal at P_0 of the curves (1) and (2) being chosen as the axes of ξ and y, the difference between the ordinates of the curves (1) and (2) is $A\xi^2 + B\xi^3 + \cdots$, where

$$A = \tfrac{1}{2}\,\frac{\sin 2\phi_0}{\rho_0}\left\{ 8\,\frac{g_1^2}{g_2^3}\,\rho_0\omega^2 - \frac{g_1}{g_2^3}\,\rho_0^2\omega^4(8\cos^2\phi_0 + 3\sin^2\phi_0)\right\},$$

in which ϕ_0 is geocentric latitude of P_0, ρ_0 is ρ for P_0, g_1 and g_2 are accelerations at P_0 due to the fields F_1 and F_2 respectively. Approximately $g_1^2/g_2^3\rho_0\omega^2 = 1/(17)^2$ and $g_1/g_2^3\rho_0^2\omega^4 = 1/(17)^4$. The expression gives the southerly deviation when ξ is the height of P_0.

7. Professor Dean considers a special case of the problem in elasticity known as the problem of Clebsch, the forces being parallel to the faces of the plate and distributed continuously across the edges. The assumption of Clebsch, that certain shearing stresses are zero, restricts the solution to the case of a plate of infinitesimal thickness and is not made here. It is shown in the standard treatises that the strain perpendicular to the planes of the faces is a linear function of the coordinates parallel to the faces. From this fact and the generalized form of Hooke's law it is shown that the cubical expansion is a linear function of two variables, and that the shears are proportional to the distance from the mid-section of the plate parallel to the faces. This function substituted in the displacement equations gives two Poisson equations by means of which the displacements are expressed as logarithmic potential functions. The stresses being functions of the derivatives of the displacements are easily determined in terms of the coordinates and three arbitrary constants, which may be determined from the boundary conditions of stress. The paper will be offered to the *Philosophical Magazine*.

8. Professor Davis shows that, taking two vectors as representative of two imaginary points, it is possible to pass through them an infinite number of imaginary lines. This is effected by proper projective changes in the system of coordinates. It is shown also that lines in different systems may have an infinite set of elements in common. The paper will be embodied in one to appear in the *Nebraska University Studies*.

9. In Professor Westfall's paper the continuity and boundedness of integral rational functions of infinitely many variables in a domain D_m: $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$, is discussed. If $\lim f(x) = f(a)$ whenever $\lim \sum (x_i - a_i)^2 = 0$, for one point a of D_m, this holds for every point a, and the integral rational function is bounded in any given domain D_m. If $\lim f(x) = f(a)$ when $\lim x_i = a_i (i = 1, 2, 3, \cdots, + \infty)$ when $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$ and $\sum_{i=1}^{i=\infty} a_i^2 < m$, this again holds true for any point x in any domain D_m. That is, semicontinuity or continuity at a single interior point is sufficient to ensure the same property at every point of any domain D_m in the case of integral rational functions.

10. In a paper presented at the summer meeting of the Society at Princeton, September 14, 1909, Professor Newson announced a fundamental theorem in the theory of linear groups. His present paper is devoted to the application of this theorem to the theory of linear groups in two variables. He is able to find all the continuous groups as previously determined by Lie, and all of the discontinuous groups as determined by Klein and also by Gordan.

11. By means of the known relation $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ Professor Brenke obtains a linear differential equation in S_k, where S_k denotes the sum of the first k terms of the series. On solving for S_k and passing to the limit there results the equation

$$\sum_1^\infty c^n J_n(x) = \tfrac{1}{2} e^{ax} \int_0^x [cJ_0(x) + J_1(x)] e^{-ax} dx \,; \quad \left[a = \frac{c^2 - 1}{c} \right].$$

The result holds for all values of x and c.

12. Fréchet proved, in his thesis, that the Heine-Borel theorem holds for countable families in any compact assemblage in which "limit" is definable by means of "distance" in the sense in which those terms are there used. Professor Hedrick points out in the present paper that the same theorem is true for any compact, closed assemblage whose derivative is closed.

13. Mr. Frizell defines two rules of combination of symbols,

— a higher rule distributive according to the inductive formulas

(1) $(a \circ \mu)b = ab \circ \mu b, \cdots,$ (2) $a(b \circ \mu) = ab \circ a\mu, \cdots,$

over a lower rule associative according to the inductive formula

$$(3) \qquad a \circ (b \circ \mu) = a \circ b \circ \mu, \cdots;$$

postulates two classes of symbols both possessing the fundamental group property as regards the rule denoted by the sign $\circ$, of which (4) one shall contain the symbol μ, and (5) the other shall contain the symbol $\mu\mu$, where (6) $\mu_i \circ \mu_j \neq \mu_i \cdots$; and demands (7) that the two classes shall be identical.

It follows that the well ordered infinite class defined by the above seven postulates constitutes an abelian semigroup with regard to each rule of combination and has a modulus for the higher rule, which is distributive over the lower.

14. In Dr. Ingold's paper an effort is made to determine how far arbitrary relations may be assigned between a tangent vector T (supposed to be a function of a parameter s) and its derivatives T', T'', T''', $\cdots$. A set of normal vectors N_1, N_2, N_3, $\cdots$ are defined by the relations

$$N_i = A_{i0}N_0 + A_{i1}N_1 + \cdots + A_{ii-1}N_{i-1} + A_{ii}T^{(i)},$$

where the scalar coefficients A_{ij} are so defined that the N_i form an orthogonal set of unit vectors.

It is shown first that the A_{ij} are expressible in terms of i quantities $1/r_i$ and their derivatives; that the $1/r_i$ are the curvatures of the curve thus defined; finally that the A_{ij} certainly exist if T is explicitly given, since the r_i, and hence also the A_{ij}, can be expressed in terms of T and its derivatives.

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