# INFINITE SERIES.

*Lehrbuch der unendlichen Reihen.* Von DR. NIELS NIELSEN. Vorlesungen gehalten an der Universität Kopenhagen. Leipzig, Teubner, 1909. viii + 287 pp. Price 12 M.

THIS excellent elementary textbook shows the careful working over which is necessary to make a course of lectures fit the requirements of the classroom. From the beginning to the end the author constantly has in mind the students before him. The foundations do not presume a partial structure already in the students' minds, but start from the bare fundamentals all must possess to understand the course at all. For this reason the work is divided into three parts : theory of sequences, series with constant terms, series with variable terms. We may say that the first part defines what a series must mean, and what it can give us ; the second part discusses the management of the particular value of a series for a given value of the variable ; the third part discusses the sweep of values due to different values of the variable. The development proceeds leisurely and is well illustrated with examples. The references are sufficiently numerous to incite the student to follow up the subject in original papers, but not so exhaustive as to overwhelm him.

The first part contains six chapters, in order : rational numbers, irrational numbers, real sequences, complex sequences, applications to elementary transcendental functions, doubly infinite sequences. The conception of rational numbers and their combinations under the four processes of arithmetic is the beginning. From this is developed immediately the idea of rational sequence, and limit. It is then shown that any periodic decimal fraction represents a rational number, and that non-periodic decimals represent limits but not rational numbers. An irrational number $a$ is defined to be the non-rational limit of two approximation series $h_n$, $l_n$, such that

$$h_n > a > l_n, \quad \lim (h_n - a) = 0, \quad \lim (a - l_n) = 0.$$

The irrational $\omega$, $h_n > \omega > l_n$, is greater than the irrational $\omega'$, $h'_n > \omega' > l'_n$, if for a certain $n$, $l_n > h'_n$ ; and $\omega$ is less than $\omega'$, if for a certain $n$, $h_n < l'_n$. In any other case, for each $n$, $h_n \geqq l'_n$ and $h'_n \geqq l_n$ and $\omega = \omega'$. The four fundamental operations are defined for irrationals by defining the limits of $h_n \pm h'_n$, $l_n \pm l'_n$, or $h_n h'_n$, $l_n l'_n$, $h_n : h'_n$, $l_n : l'_n$, respectively. With these definitions it is shown that every irrational has an incommensurable decimal expression, and that irrational sequences lead to no new varieties of number. Thus the rational sequence really closes the number system. The development followed here is a special case of that of G. Cantor.*

Sequences of real numbers are classified as convergent or fundamental series; divergent, which are either proper or improper according as the sequence from a certain term on retains the same sign or not; and oscillating. The limits of the sum, difference, product, and quotient of two fundamental series are shown to be the sum, difference, product, and quotient, respectively, of the two limits. The general test of convergence is proved in these terms: †

*The necessary and sufficient condition for the convergence of the sequence $a_0, a_1, \cdots, a_n, \cdots$ is that for any preassigned arbitrary small positive number $\epsilon$ there exists a positive integer $N$, such that for $n \geqq N$*

$$| a_{n+p} - a_n | < \epsilon$$

*for any positive integer $p$.*

The author now introduces the conceptions of monotone sequence, of limit points, of upper and lower boundaries, of superior and inferior limits. He closes the chapter with a theorem of Abel: ‡

*If there is given a monotonic non-increasing sequence $\epsilon_n$, and if from a given sequence $\cdots, a_n, \cdots$ we form a new sequence $\cdots, s_n, \cdots$, where $a_p = s_p - s_{p-1}$, then for every $n$*

$$\epsilon_1 \cdot g_n \leqq a_1 \epsilon_1 + \cdots + a_n \epsilon_n \leqq \epsilon_1 \cdot G_n,$$

where $G_n$ and $g_n$ are respectively the upper and lower boundaries of the sequence $s_n$.

In the next chapter, on complex sequences, the complex number is treated as a couple $(a, b)$. The usual formulas of combination are developed, and the convergent complex sequence is defined to be one such that the sequence of the moduli of its terms is convergent. A theorem of Jensen,§ which is a generalization of one of Cauchy, is given:

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* *Math. Annalen*, vol. 5 (1875), p. 128.

† Du Bois-Reymond: Allgemeine Funktionentheorie, vol. 1, p. 6, 260.

‡ *Crelle*, vol. 1 (1826), p. 314.

§ *Tidsskrift for Mathematik*, ser 5, vol. 2. (1884), pp. 81–84. *Comptes Rendus*, vol. 106 (1888), pp. 833–836.

*If the sequence* $\cdots, a_n, \cdots$ *satisfies the two conditions*

$$(1) \qquad \lim_{n \doteq \infty} |a_n| = \infty,$$

*and for every n*

$$(2) \quad |a_1| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}| < B.|a_n|,$$

*where B is a fixed determinate number independent of n; if moreover the funaamental series* $\phi_1, \phi_2, \cdots \phi_n, \cdots$ *has the finite limit* $\Phi = lim_{n \doteq \infty} \phi_n$, *then we also have*

$$\Phi = \lim_{n \doteq \infty} \left( \frac{a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n}{a_n} \right).$$

From this we may deduce the formula

$$\lim_{n \doteq \infty} \left( \frac{b_n}{a_n} \right) = \lim_{n \doteq \infty} \left( \frac{b_n - b_{n-1}}{a_n - a_{n-1}} \right),$$

where $b_n = a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n$.

For $a_n = n$, we have a theorem of Cauchy :*

*If in the sequence* $\cdots, b_n, \cdots \ lim_{n \doteq \infty} (b_n - b_{n-1})$ *is finite and determinate*, *then*

$$\lim_{n \doteq \infty} \left( \frac{1}{n} b_n \right) = \lim_{n \doteq \infty} (b_n - b_{n-1}).$$

In the second part, which treats of real series, a series is defined to be a sequence $\cdots, s_n, \cdots$ of sums of terms $u_1, \cdots, u_n, \cdots$. Convergence of a series is distinguished from summability. If the new sequence $S_n = (s_1 + s_2 + \cdots + s_n)/n$ is formed and if it is a convergent sequence, then the series $u_1, u_2, \cdots, u_n$, which gives $\cdots, s_n, \cdots$ is said to be summable. A generalization of this is also suggested, through the Jensen theorem mentioned above. Convergence which is independent of the order of the terms is unconditional; otherwise it is conditional. Absolute convergence exists when the series of absolute values of the terms converges. It is shown that unconditional convergence and absolute convergence imply each other.

Certain cases of convergence are now considered. These are: the well-known theorem of Leibniz on alternating series;

* Analyse algébrique, 1821, p. 54.

comparison series; a theorem of Pringsheim to the effect that *if* $\cdots, a_n, \cdots$ *is monotone increasing and divergent, then the series whose terms are*

$$u_n = \frac{a_{n+1} - a_n}{a_{n+1} \cdot a_n^\rho}$$

*is convergent for every positive* $\rho$; logarithmic tests; a theorem of Jensen: *

*If* $\lim_{n=\infty} | a_n | = \infty$, *and* $| a_1 | + | a_2 - a_1 | + \cdots + | a_n - a_{n-1} | < B \cdot | a_n |$, *but otherwise the values of* $a_n$ *are quite arbitrary, then*

$$\sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{a_n}$$

*is divergent;* several conclusions are drawn from this, embodying theorems of Dini; two theorems ascribed to both Du Bois-Reymond † and Dedekind ‡:

1. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *are two infinite series of arbitrary complex numbers which satisfy only the conditions that both*

$$\sum_{n=0} a_n \text{ and } \sum_{n=0}^{\infty} | b_n - b_{n+1} |$$

*converge, then* $\sum a_n b_n$ *converges.*

2. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *satisfy only the conditions* $\lim_{n=\infty} b_n = 0$, $\sum | b_n - b_{n+1} |$ *converges, and* $\sum a_n$ *oscillates between finite boundaries, then* $\sum a_n b_n$ *converges.*

The chapter following is devoted to elementary tests of convergence. This seems to be a purely artificial separation of the subject treated in this chapter and the preceding. The tests given are (1) Kummer's: §

*The series of positive terms* $\sum u_n$ *is convergent if it is possible to determine a series of positive terms* $\cdots, \phi_n, \cdots$ *such that for* $n \geqq N$

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} \geqq \alpha > 0.$$

*Likewise divergent if*

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} < 0.$$

* *Tidsskrift for Mathematik* (1884), ser. 5, vol. 2, p. 85.

† Neue Lehrsätze über die Summen unendlicher Reihen, p. 10.

‡ Dirichlet, Vorlesungen über Zahlentheorie, § 101, 3 Aufl., 1879.

§ *Crelle*, vol. 13 (1835), p. 172.

(2) Cauchy's: *

*The series of positive terms* $\sum u_n$ *is convergent or divergent according as*

$$\lim_{n \doteq \infty} \inf \frac{u_n}{u_{n+1}} > 1 \quad or \quad \lim_{n \doteq \infty} \sup \frac{u_n}{u_{n+1}} < 1.$$

(3) Duhamel's: †

*If for the series of positive terms* $\sum u_n$ *we have a development of the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1+\alpha}{n} + \cdots,$$

*the series converges or diverges according as* $\alpha > 0$ or $\alpha < 0$.

(4) Raabe's: ‡

*If for the series of positive terms* $\sum u_n$ *we have*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{a}{n^{1+\delta}} + \cdots,$$

*where* $a$ *is finite and* $\delta$ *is an arbitrary small but assignable positive number, then* $\sum u_n$ *diverges.*

(5) The logarithmic test.

Following these is a chapter on the multiplication of series, which is given a somewhat original treatment.

After an excellent chapter on transformation of series for more rapid numerical computation, the question of convergence is again taken up. The preceding theorems are widened and generalized and the discussion is led to the general criteria of Pringsheim. From his criteria of the first and second kinds are shown to follow all the more elementary criteria. It seems unnecessary to trace the development further. It is evidently along teachable lines, and will give the student a clear understanding of the question of convergence. This part is closed with a treatment of infinite products, infinite continued factors, and double series.

The last part treats of series of variable terms. In the first chapter the idea of point set (Menge), is developed, with its usual accompanying terms. In the second chapter the idea of

* Analyse algébrique, p. 134.

† *Journ. de Math.*, vol. 4 (1839), pp. 214–221.

‡ *Journ. von Ettinghausen und Baumgartner*, vol. 10 (1832); *Crelle*, vol. 11, pp. 309–311. *Journ. de Math.*, vol. 6 (1841), pp. 85–88.

function of one or two real variables, and of a complex variable is developed. The function is handled as a point set of functional values. The general principle of convergence is stated in these terms :

*The necessary and sufficient condition for the existence of* $f(a-0)$, *and thus for the existence of a limit* $A$ *for the function* $f(x)$ *as* $x$ *approaches* $a$, *is that it be possible to find for any preassigned arbitrary small number* $\epsilon > 0$ *a number* $\delta > 0$ *such that if* $a-\delta \leqq x \leqq a$, $a-\delta \leqq y \leqq a$, *then* $|f(x)-f(y)| < \epsilon$.

A similar condition is stated for a function of a complex variable. The definition of, and conditions for, continuity follow. It is shown that any rational function of continuous functions of $x$ is continuous, save for those values which make the resulting denominator vanish. Also if $y=f(x)$ is continuous for $x=a$, and $z=\phi(y)$ is continuous for $y=b=f(a)$ then $z=F(x)=\phi[f(x)]$ is continuous for $x=a$. The existence of maxima and minima is then proved as usual.

The next chapter treats of uniform convergence. Seven theorems are given to the effect that

1. *A convergent sequence of continuous functions of* $x$ *which is uniformly convergent in a given interval defines a continuous function of* $x$ *over this interval.*

2. *An infinite convergent series is uniformly convergent over an interval if the remainder after the nth term is convergent independently of the values of* $x$ *in the interval.*

3. *An infinite convergent series of continuous functions of* $x$ *which is uniformly convergent over an interval defines a continuous function of* $x$.

4, 5, 6, 7. These relate to the convergence of infinite products.

To illustrate these theorems the gamma function is briefly treated. The following chapters discuss trigonometric series, power series, Dirichlet series, and faculty series. In the latter chapter Professor Nielsen has incorporated some original work. The 210 problems are very carefully selected to throw light on the developments which they follow. The text as a whole is rather carefully graded as to difficulty and yet is full enough for a first course. Some few errors remain uncorrected but these are easily apparent to the reader.

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