The auxiliary function θ is the temperature and the obvious physical mode of solution is Liouville's method of successive substitutions.*

A case of special interest physically is that in which k is defined by

$$k\kappa = 1.$$

Is there any method of numerical computation better than approximate integration?

HAVERFORD COLLEGE,
March, 1910.

GRASSMANN'S PROJECTIVE GEOMETRY.

Projektive Geometrie der Ebene unter Benutzung der Punktrechnung dargestellt. Von HERMANN GRASSMANN. Erster Band: *Binäres.* B. G. Teubner, 1909. 8vo. xii + 360 pp.

MODERN projective geometry is two-sided. Either use is made of algebraic analysis in its development or it is developed from the fundamental concepts of point, line, plane by means of certain axioms and postulates. In the one case it is analytic, in the other synthetic. Usually the two methods of presentation are more or less combined, with the emphasis laid upon the one or the other. If the analytic method is adopted, operations are usually carried out in cartesian space with the aid of a system of coordinates. The synthetic method makes no use o coordinate systems.

Professor Grassmann's work is analytic in character in that use is made of algebraic analysis. It is unique in discarding the usual coordinate systems and adopting ideas due to Möbius and to the elder Grassmann. These ideas found expression in the Baryzentrische Calcul and in the Ausdehnungslehre.

In the last quarter century a number of writers have made use of these ideas; notably, Stéphanos, H. Wiener, Segre, Peano, Aschieri, Study, Burali-Forti. It is the author's purpose to bring the results of these writers and of others together into a connected course covering the fields of binary and ternary linear transformations. This is certainly a most worthy purpose and mathematicians will be grateful to the author for the evident care and devotion with which he has set about the performance of his task.

* Maxime Bôcher, An introduction to the study of integral equations. Cambridge, Eng., 1909.

The present volume is devoted to the binary field and is filled with detail much of which must be passed over without notice. It is furnished with a good table of contents and a register of material and names, which aid substantially in following the main lines of thought. What follows may serve to illustrate these. There are three Hauptteile or chapters. In the first of these the reader is introduced to the Punktrechnung by means of which points and lines are added, subtracted, and multiplied and the laws governing these operations formulated. Thus a point is represented by two factors, a scalar m called its mass and a vector f depending only upon its position. The sum of two points $m_1 f_1$ and $m_2 f_2$ is a point ms whose mass m is $m_1 + m_2$ and which has the position of the center of gravity of the two given points. Points of zero mass are ideal or at infinity. Points of unit mass are said to be simple points, others are called multiple points.

The process of multiplying one point a by another b is called exterior (äussere) multiplication and is indicated by the symbol $[ab]$. It is non-commutative, $[ab] = -[ba]$. The product is conceived of as a force acting along the segment from a to b and is called a Stab — a designation used by the author in 1894 (Punktrechnung und Projektive Geometrie, Halle) and since quite generally adopted by German writers. If a and b are finite points, the product $[ab]$ vanishes only when $a = b$.

The difference between two simple points is the segment (Strecke) connecting them. The point calculus will naturally be more familiar to the student of vector analysis than to one accustomed to think in terms of ordinary coordinate systems.

In the second chapter one finds an application of the point calculus to elementary projective geometry. A group of four points on a line in the order a, c, b, d, is called, after Von Staudt, a Punktwurf and its anharmonic ratio is defined as the double ratio

$$\frac{[ac]}{[cb]} \div \frac{[ad]}{[db]}.$$

This, being the double ratio of four Stäbe, on the same line, is a numeral quantity. The definition of projectively related ranges and pencils follows, together with the generation of curves of second class and of second order. The harmonic properties of the complete quadrilateral and the Pascal and Brianchon theorems are derived easily and thus we have the elementary part of projective geometry.

If the point calculus were an end in itself, or if it found its chief application in developing this elementary geometry, it would be rather more curious than useful. But the advantage comes in studying linear transformations in the n-ary field and is perhaps increasingly manifest the greater the value of n. Hence the reader's interest will be aroused and will increase as he reads on in the third and last chapter. This chapter is devoted to projectivities on a line and in a plane pencil and contains rather more than two thirds the entire volume.

For the analytic representation of the points of a point row, two non-coincident base points e_1 and e_2 are chosen whose masses are determined so that a third point e, not coinciding with either of the others, but otherwise arbitrary as to mass and position, shall be the sum of the other two. This third point is the unit point. Any point x of the point row is then given by the formula

$$x = x_1 e_1 + x_2 e_2,$$

where x_1 and x_2 are numerical quantities. A second point row on the same line whose base points and unit point are respectively a_1, a_2, and a is projectively related to the first when the base points and unit point of the one are made to correspond to the base points and unit point of the other, each to each.

So far this is not unlike the usual introduction to the study of linear transformations in the binary field. The divergence comes in the next step. A factor p* (Abbildungsfaktor) is defined so that

$$e_1 p = a_1,\ e_2 p = a_2,\ ep = a\ ;\ i.\ e.,\ (e_1 + e_2)\, p = e_1 p + e_2 p,$$

and also

$$xp = (x_1 e_1 + x_2 e_2) p = x_1 a_1 + x_2 a_2.$$

The operation p so defined may be represented formally by what the author calls an extensive fraction

$$p = \frac{a_1,\ a_2}{e_1,\ e_2}.$$

This brings into evidence the base points of the two point rows. The operation indicated by p is distributive over the sum of any number of points on the line and associative when applied

* Many of the symbols in the text are expressed by German letters.

to the product of a point by a numerical quantity. Similar considerations hold for a projectivity established between two sheaves of lines belonging to the same plane pencil of rays.

The commutative and associative laws hold for the sum of any number of symbols representing projectivities on the same line or in the same pencil. The product of two such symbols p, q is called the resultant product (Folgeprodukt) and is defined as a third projectivity on the same line obeying the associative law

$$x(pq) = (xp)q.$$

The resultant product is in general non-commutative. The method for calculating this resultant product is given, and we are led at once to the theorem that all projectivities on a line form a group.

A projectivity $p = a_1, \ a_2/e_1, \ e_2$ possesses an inverse $p^{-1} = e_1, e_2/a_1, a_2$, provided the exterior product $[a_1a_2] \neq 0$; *i. e.*, provided the base points of the second point row are not coincident. In the opposite case, p is degenerate and does not possess an inverse. If p possess an inverse, the equation

$$xp = x'$$

is solvable for x.

The combination product $[yz \cdot pq]$, where y and z are points on the line, is defined by the equation

$$[yz \cdot pq] = \tfrac{1}{2}\{[yp \cdot zq] - [zp \cdot yq]\}.$$

A number multiplying any one of the letters y, z, p, q may be written before the entire expression, and the symbol itself is distributive over a sum replacing any one of its constituents. The expression

$$\frac{[yz \cdot pq]}{[yz]}$$

is independent of the particular points y, z used in forming it. Hence

$$\frac{[yz \cdot pq]}{[yz]} = \frac{[e_1e_2 \cdot pq]}{[e_1e_2]} \equiv [pq].$$

This new symbol $[pq]$ is called the combination product of the projectivities p and q, and is always commutative in contradistinction to the resultant product.

The combination square or power of a projectivity

$$[p^2] = \frac{[e_1e_2 \cdot pp]}{[e_1e_2]} = \frac{[a_1a_2]}{[e_1e_2]}$$

is a numerical quantity, since it is the ratio of the two Stäbe connecting the base points of each point row.

A projectivity p is directly or oppositely projective; i. e., the points of the two point rows are arranged in the same or in opposite senses according as $[p^2]$ is positive or negative. If $[p^2] = 0$, then $[a_1a_2] = 0$ and the projectivity is degenerate.

A double point d of a projectivity p satisfies the equation

$$dp = rd,$$

where r is a numerical quantity, hence it is altered only as to mass by the projectivity. This equation easily transforms into a quadratic in r whose coefficients depend only upon the base points of the two point rows. The roots of this equation are termed the principal numbers of the projectivity and the equation itself its principal equation.

If a projectivity leaves every point of the line unaltered as to mass and position, it is identity. If it alters the mass only, it is a coincidence (Deckung). If the sum of the principal numbers is zero, the projectivity is an involution. The properties of the involution follow together with an application to the vector equations of the ellipse and the hyperbola. The equation of the ellipse, for example, comes out to be

$$x = a(\cos w + \mathbf{e} \sin w) = ae^{\mathbf{e}w},$$

where x is any point of the curve, a the semi-major axis, w the eccentric angle, e the Naperian base, and $\mathbf{e}$ the elliptic involution which changes one set of diameters into the conjugate set.

The consideration of projectivities with conjugate imaginary principal numbers is introduced by a study of what are called positive and negative circular forms (Abbildungen), after a designation due to the elder Grassmann. A positive circular form in the plane pencil is a projectivity transforming a sheaf of lines into a congruent sheaf, i. e., a rotation. A positive circular form upon a line is a section of the congruent sheaves by a line not passing through their common center. Every projectivity with conjugate imaginary principal numbers is identical geometrically to a positive circular form and can differ from it analytically only by a numerical factor.

A negative circular form in the plane pencil transforms a sheaf into a congruent sheaf, the rays of the two sheaves being arranged in opposite senses; i. e., it is a reflection across one of the double rays of the projectivity. A section by a line not passing through the common center gives rise to a negative circular form upon that line.

A circular form is expressible exponentially in terms of a certain involution. For example, the positive circular form on a line is

$$C_{(a,\, b,\, w)} = e^{\mathsf{e}\, w},$$

where a, b are the base points of the first point row, w is a real parameter, e the Naperian base, and e is the elliptic involution

$$\mathsf{e} = \frac{b, \, -a}{a, \, b}.$$

For a negative form, the involution in the exponent is hyperbolic

$$h = \frac{b, \, a}{a, \, b}.$$

The double points of these involutions coincide with the double points of the corresponding circular form.

All positive circular forms with the same point pair a, b form a continuous one-parameter group. Negative circular forms do not form a group. The totality of all positive and negative circular forms with the same point pair a, b forms a discontinuous group which contains the continuous group of positive circular forms as a subgroup.

The circular forms are not the only projectivities expressible exponentially in terms of an involution. The consideration of linear systems of projectivities, in particular a sheaf

$$lp + mq,$$

where l and m are numerical quantities, leads to the fact that, unless all the projectivities of the sheaf are involutions, there is but one involution contained in the sheaf. If the projectivity q is fixed, say identity, the double points of the involution contained in the sheaf coincide with the double points of p, and p is expressible in terms of the identity and this involution double points.

A direct (gleichläufig) projectivity with real and distinct double points is then

$$p = e^{hw},$$

where h is the double point involution and w a real parameter. On the other hand an opposite (gegenläufig) projectivity expressed in terms of its double points involution is

$$q = he^{hw}.$$

Group properties follow as in the case of the circular forms.

Two projectivities for which the combination product $[pq]$ vanishes are called harmonic. The name is due to Segre as well as many of their properties (*Crelle*, volume 100).

The domain of all the projectivities on a line is defined by means of four unit fractions (Ausdehnungslehre)

$$\mathrm{e}_{11} = \frac{e_1, 0}{e_1, e_2}, \quad \mathrm{e}_{12} = \frac{e_2, 0}{e_1, e_2}, \quad \mathrm{e}_{21} = \frac{0, e_1}{e_1, e_2}, \quad \mathrm{e}_{22} = \frac{0, e_2}{e_1, e_2}.$$

These are all degenerate projectivities, the second and third being parabolic involutions, and hence the power of each vanishes. Any projectivity

$$p = \frac{a_1, a_2}{e_1, e_2}$$

can be expressed as a linear function of these four units. For if we put

$$a_1 = a_{11}e_1 + a_{12}\, e_2, \quad a_2 = a_{21}e_1 + a_{22}e_2,$$

then

$$p = a_{11}\mathrm{e}_{11} + a_{12}\mathrm{e}_{12} + a_{21}\mathrm{e}_{21} + a_{22}\mathrm{e}_{22}.$$

The proof consists in applying p, thus expressed, to each of the base points e_1, e_2 and showing that they transform into a_1, a_2 respectively.

The four units are linearly independent. For the assumption of an equation of the form

$$c_{11}\mathrm{e}_{11} + c_{12}\mathrm{e}_{12} + c_{21}\mathrm{e}_{21} + c_{22}\mathrm{e}_{22} = 0$$

leads to the vanishing of all the c's since e_1 and e_2 are distinct points.

The above representation of p is unique. For if p could be expressed by a set of a's and also by a set of b's, the difference be tween the two expressions would vanish and thus the a's equal the b's, each to each.

Of course any four linearly independent projectivities may be taken for a base. It is convenient to choose four which are mutually harmonic; these are, when expressed in terms of the unit fractions,

$$1 = e_{11} + e_{22}, \quad h_1 = e_{11} - e_{22}, \quad h_2 = e_{12} + e_{21}, \quad e = e_{12} - e_{21},$$

where 1 is identity, h_1 and h_2 are hyberbolic involutions, and e is an elliptic involution. It is easily shown that they are mutually harmonic by means of the laws governing the multiplication of unit fractions and the condition which two projectivities must satisfy in order to be harmonic. Any projectivity on the line is then

$$p = a + a_1 h_1 + a_2 h_2 + a_3 e.$$

This expression of p leads at once to the Stéphanos (*Mathematische Annalen*, volume 22) representation of the projectivities on a line by the points of ordinary space; viz., the images of the four fundamental projectivities 1, h_1, h_2, e are taken to be the vertices of a fundamental tetrahedron whose unit point is arbitrary. The image of any projectivity p is then the point whose homogeneous coordinates referred to this tetrahedron are a, a_1, a_2, a_3.

The images of all degenerate projectivities fill the quadric

$$a^2 - a_1^2 - a_2^2 + a_3^2 = 0$$

with respect to which the fundamental tetrahedron is self-conjugate. Two harmonic projectivities are imaged upon the points which are conjugate with respect to this quadric. The involutions on the line are imaged upon the points of the plane determined by the images of the three fundamental involutions h_1, h_2, and e.

A number of details of this representation are given together with a discussion of the representation of the involutions on the line by the points of a plane.

The projectivities upon a line form a system of higher complex quantities since they satisfy the four conditions laid down by Study (*Göttinger Nachrichten*, 1889). On comparing the multiplication table of the four fundamental projectivities with the multiplication table of the four units 1, i, j, k of the Hamilton quaternion system, it follows that any projectivity may be expressed as a complex quaternion, viz.,

$$p = a + a_1\sqrt{-1}i + a_2\sqrt{-1}j - a_3 k.$$

A theorem called Study's theorem (Cf. Encyclopädie, I. 1) brings these results together:

The group of all the projectivities on a line forms a system of *higher complex quantities of quaternion type. It possesses another real form which is like the system of ordinary quaternions and these two real forms are the only ones in which systems of higher complex quantities of quaternion type can appear.*

The work is manifestly a labor of love. An interesting circumstance in connection is the fact that this first volume was brought out on the hundredth anniversary of the birthday of the author's father.

L. WAYLAND DOWLING.

LINEAR DIFFERENTIAL EQUATIONS.

Vorlesungen über lineare Differentialgleichungen. Von LUDWIG SCHLESINGER. Leipzig and Berlin, Teubner, 1908.

IT is over ten years ago that the author of the present "Vorlesungen" completed the publication of his well-known Handbuch der Theorie der linearen Differentialgleichungen. As every one familiar with the older book well knows, it was intended to be, as its name implied, a handbook containing a complete treatment of all that was at that time known about the subject. It seemed natural therefore, to expect under the title of "Vorlesungen" a briefer version of the same subject, adapted to the needs of the younger student and rendered more palatable for him by a proper selection of topics and by a more elementary treatment. And in a certain sense the "Vorlesungen" may indeed be considered as an introduction into the theory of linear differential equations, in so far at least as all of the most important results of the theory built up by Fuchs and his successors are discussed. But the method of treatment is so novel and the artistic unity of the book is preserved to such an extraordinary extent that we must look upon it as an important addition to analysis rather than as a treatise of more or less pedagogical merit.

It is well known that Riemann's discussion of the hypergeometric function furnished Fuchs with the fundamental ideas which led to the modern theory of linear differential equations, which theory may be said to date from Fuchs's paper of 1865. But we now know that Riemann himself had intended to construct a general theory of linear differential equations upon the same general principles which had led to such brilliant results in