SHORTER NOTICES.

Theories of Parallelism, an historical Critique. By W. B. FRANKLAND. Cambridge, 1910. xviii + 70 pp.

AFTER reading The Story of Euclid (1901), and particularly after reading Euclid Book I with a Commentary (1902), by the same author, every student of the history of mathematics will welcome this little "historical critique," feeling sure of finding much food for thought in small compass. For it is one of the characteristics of Mr. Frankland that he says what he has to say in the fewest words possible, and hence in a book of less than a hundred pages he condenses matter that most writers would expand to fill double the space. And when we come to consider that in these few pages he has presented a scholarly digest of the theories of upward of forty geometers we begin to realize the thought that has been given to the subject and the skill that the author has shown.

Mr. Frankland gives a setting for the historical discussion in an introduction of eighteen pages. In this he begins by stating one existence assumption and one fundamental theorem as follows: "Let us assume that *straight lines* are freely applicable to themselves and to one another; and that there is a *plane* in which they are freely movable; and let us investigate the *parallelism* of such straight lines in an even plane." The theorem is that of Hilbert, that the area of any polygon is proportional to its divergence, that is, to the difference between its angle sum and $(n - 2)\pi$. From this theorem, of which the usual proof is given, the author proceeds to prove that for the elliptic, parabolic, and hyperbolic hypotheses, respectively, there are no, one, and two parallels to a given straight line, through any given point.

The historical sketch is then introduced by a discussion of Euclid's own theory, a discussion that has, of course, been anticipated by Dr. (now Sir Thomas) Heath's monumental work on the Elements. Heath calls attention to the fact that Euclid assumes the infinitude of space, and that the possibility of a straight line as re-entrant seems never to have occurred to him. He thus bars from his theory the possibility of the non-existence of parallelism. On the other hand the fifth postulate bars the possibility of double parallelism, so that there is left to him only the parabolic hypothesis. The weakest feature of his theory is that the statement of the fifth postulate positively invites attempts at proof, and, as we know, these attempts were constantly being made for over two thousand years.

Of the successors of Euclid, Posidonius (B. C. 80) used the equidistant definition, thus barring out the elliptic and hyperbolic hypotheses. Geminus (B. C. 70) has often been made to bear the blame of the definition of one Aganis, that equidistants "are such as lie in one surface, and when produced indefinitely have one space between them, and it is the least line between them." Mr. Frankland follows Dr. Heath in believing this Aganis to have been a writer about A. D. 500. Ptolemy (A. D. 150) wrote a tractate on the fifth postulate, giving four propositions by way of proof. Of these the first is substantially as follows: "If two straight lines are crossed by a transversal so that the interior angles on the same side are together equal to two right angles, then the lines can never intersect." The proof consists in showing that if they meet at O they must also meet on the other side of the transversal at O'. The possibility that O and O' may be identical, as they are in the elliptic hypothesis, evidently did not occur to Ptolemy. Proclus (A. D. 450) seems to have been the first writer to have had any of the modern view of the nature of parallelism. As Mr. Frankland says, his words might almost have been written by Lobachevsky or Bolyai. Without entering into the geometric discussion, one passage is worthy of special note. Proclus says, "It cannot be asserted unconditionally that straight lines produced from less than two right angles do not meet. It is of course obvious that some do meet, but the (euclidean) theory would require all such to intersect. But it may be urged that as the defect from two right angles increases, the straight lines continue asecant up to a certain magnitude of the defect, and for a greater magnitude than this they intersect." It is the last phrase that is significant. The next noteworthy writer on the subject was Nasr-Eddin (A. D. 1250), whose attempt is set forth in the works of Wallis. His effort is well known, tacitly begging the question by another postulate as difficult as that of Euclid. Other writers of more or less prominence are Anaritius (A. D. 900), Gerbert (A. D. 1000), Billingsley, who edited the first English edition of Euclid (1570), Clavius (1574), Oliver (1604), Savile (1621), Tacquet (1654), and Hobbes (1655). In 1663 Wallis gave his proof, or rather his well-known substitute for the postulate. This substitute is assumed "as an universal idea: To any given figure whatever, another figure, similar and of any size, is possible." Leibnitz

(1679) seems to have had "logical premonition of the elliptic hypothesis" in the note that he gives on the definition of Euclid.

Coming to what may be called the modern school, the name of Saccheri (1733) naturally stands out as chronologically the first. His theory is too well known through the work of Engel and Stäckel to need any description in a review, but its prime weakness lay in the failure to consider the possible curvature of three-dimensional space. Simson's (1756) substitute for Euclid's postulate is well known as an educational rather than a mathematical effort. Lambert's (1766), however, is quite the reverse, and essentially he recognizes the three possible hypotheses which Klein finally named the parabolic, elliptic, and hyperbolic. Bertrand's (1778) theory is less familiar, and it has all the charm of style that characterized this writer, but it has not the breadth of view of Lambert's, nor indeed of Saccheri's. Playfair's (1795) adaptation of the proposition of Proclus is well known, since it is the postulate of parallels of our ordinary textbooks, and is noteworthy as an educational measure. The relations of Gauss to the Bolyais are fairly stated, and the hypothesis of the latter is set forth in a clear fashion. To Lobachevsky, however, Mr. Frankland, following Dr. Whitehead and others, gives the greatest praise, both in the matter of priority and of clearness. Such of the modern writers as have contributed to the theory, including Riemann, Cayley, Beltrami, Clifford, and Klein, are mentioned, thus bringing the "critique" up to the present time. Perhaps a quotation from Clifford, that may not have come to the attention of all readers, may be permitted, even at the risk of extending a review already too long: "I hold in fact: (1) That small portions of space are of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them. (2) That this property of being curved or distorted is continually passed on from one portion of space to another after the manner of a wave. (3) That this variation of the curvature of space is what really happens in that phenomenon which we call the motion of matter whether ponderable or ethereal. (4) That in the physical world nothing else takes place but this variation, subject, possibly, to the law of continuity." As Frankland remarks, "the boldness of this speculation is surely unexcelled in the history of thought."

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