SHORTER NOTICES.

The Fundamental Laws of Addition and Multiplication in Elementary Algebra. By EDWARD V. HUNTINGTON, Harvard University. Reprinted from the *Annals of Mathematics*, Second Series, Volume 8, No. 1 (October, 1906). Publication Office of Harvard University. 44 pp.

THE object of this paper, as stated in the introduction, is "to present a list of fundamental propositions for algebra, from which, on the one hand, all the other propositions of algebra can be deduced, and in which, on the other hand, no superfluous items are included, — a list, in short, which is sufficient and free from redundancies."

The propositions are stated in terms of abstract undefined symbols a, b, etc., and two combinations of these symbols denoted by $\oplus$ and $\odot$. More briefly stated, the question which the paper answers is : "Given a class of elements with two rules of combination, what conditions must such a system satisfy in order to be formally equivalent to one of the systems of ordinary algebra?"

The first conditions imposed are the ten laws :

A_1. $a \oplus b$ is a unique element of the system.
A_2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
A_3. (1) If $a \oplus x = a \oplus y$ then $x = y$.
(2) If $x \oplus a = y \oplus a$ then $x = y$.
A_4. If $\mu x = \mu y$ then $x = y$, μ being an ordinary integer.
A_5. $[a \oplus b = b \oplus a]$.
M_1. $a \odot b$ is a unique element of the system.
M_2. $(a \odot b) \odot c = a \odot (b \odot c)$.
M_3. (1) If $a \odot x = a \odot y$ and $a \oplus a \neq a$ then $x = y$.
(2) If $x \odot a = y \odot a$ and $a \oplus a \neq a$ then $x = y$.
M_4. (1) $a \odot (b \odot c) = (a \odot b) \oplus (a \odot c)$.
(2) $(b \oplus c) \odot a = (b \odot a) \oplus (c \odot a)$.
M_5. $a \odot b = b \odot a$.

Emphasis is placed on the purely abstract character of these symbols and on the formal character of the deduction from them. "How can we be sure that our deduction is rigorous? . . . The only way to avoid the danger of using in our reasoning other properties besides those expressly stated in the fundamental assumptions is to regard them not as axiomatic propositions about numbers, but as *blank forms* in which the letters a, b, c, etc. may denote any objects we please and the symbols $\oplus$ and $\odot$ any rules of combination ; . . . The deduction from such blanks must necessarily be purely formal, and hence will not be affected by the troublesome connotations which would be sure to attach themselves to any concrete interpretation of the symbols." This we recognize of course as the general program of modern work in the foundations of mathematics.

The independence or non-redundancy of the system is exhibited in the usual manner, by showing concrete systems in which all but one of the assumptions are verified. Thus A_2 is shown to be independent of the other nine laws by the following systems: "Let the class considered be the class of all rational numbers. Let $a \oplus b = 2(a + b)$ and $a \odot b = ab$."

The following argument is given to show that A_5 is a consequence of the remaining nine of the ten fundamental postulates:

$$(a + b)(c + c) = (a + b)c + (a + b)c$$
$$= ac + bc + ac + bc \qquad (M4_1,\ M4_2),$$

also

$$(a + b)(c + c) = a(c + c) + b(c + c)$$
$$= ac + ac + bc + bc \qquad (M4_2\ M4_1).$$

Hence

$$ac + ac + bc + bc = ac + bc + ac + bc$$

and

$$bc + ac = ac + bc \qquad (A3_1\ A3_2).$$

Hence

$$(b + a)c = (a + b)c \text{ and } b + a = a + b \quad (M4_2\ M3_2).$$

The consistency of the fundamental laws is shown by giving a concrete example which is assumed to be self-consistent in which all the postulates are satisfied. The example chosen is the Argand diagram representing the complex numbers, in reality a vector field. It should be noted that this by no means proves absolutely the consistency of the postulates, but merely transfers the question to another realm. The consistency of each of the many special algebras developed in the paper follows from the fact that it is a subalgebra of this most general algebra employed.

It is shown that many essentially different systems satisfy these ten postulates. Thus they are all satisfied by the numbers of ordinary algebra with addition and multiplication retaining their usual meaning, and also by a system in which $a \oplus b = a + b + 1$ and $a \odot b = ab + (a + b)$. This gives rise to a discussion of isomorphism and categoricality. Two systems satisfying these general laws are said to be isomorphic with respect to addition and multiplication when the following conditions are satisfied:

1) the elements of the two systems can be brought into one-to-one correspondence $\cdots$ and

2) this correspondence can be set up in such a way that whenever a and b in one class corresponds to a' and b' in the other class, then $a + b$ will correspond to $a' + b'$ and $a \times b$ will correspond to $a' \times b'$.

If a set of postulates is such that any two systems which satisfy it are isomorphic, then the set of postulates is said to be categorical.

While the ten postulates given above do not therefore form a categorical system, special laws may be added to them in various ways to form categorical systems. Thus by adding the postulates

E_1. There is a unit element in the system.

F. There are no elements in the system besides those required by the other postulates.

It is then asserted that the algebra of positive integers is completely determined by the postulates A_1, 2; M_1, 2, 3, 4; E_1; F. In a paper avowedly abstract and purely deductive this seems a somewhat daring statement. It is proved, it is true, that these positive integral elements form a closed system with respect to addition and multiplication, but is this equivalent to showing that the set of postulates just given is categorical according to the definition above? The truth seems to be that we are certain our system is as categorical as some other systems to which we may prove it equivalent. Again, as with the question of consistency, the difficulty has simply been transferred to another realm.

Many other categorical systems are obtained in a manner similar to the above by adding special assumptions to those already given.

It should be remarked that in order to form an independent system each of the ten fundamental postulates except A_1 and M_1 must be regarded as making the existence of the sums and products hypothetical. Thus A_2 must read, if stated fully, "If $(a \oplus b) \oplus c$ and $a \oplus (b \oplus c)$ exist, etc."

The original name for the unproved propositions of a mathematical science was "axiom," — a truth so simple that everyone must assent to it whenever the statement is fully comprehended. In this respect the point of view has changed completely. If a, b, $\oplus$, $\odot$ are purely abstract symbols, then no proposition whatever is evident about them. Hence the word "axiom" with its old connotation is being discarded. The paper under review uses "postulate." Other writers, as Veblen and Young, are using "assumption."

This paper should serve two distinct and very useful purposes. The writer of elementary algebras for college use will have at hand a set of postulates which will serve directly as a basis for much of his work and as a model which we hope may guide his way in making the extensions necessary to characterize the complete algebra. This should render less prevalent in the future the numerous logical incongruities that so often have marred otherwise excellent texts.

The subject with which the paper deals is confessedly abstract but the style is so lucid and the mode of treatment so simple that it should be within the reach of students even in the first years in college. It is the feeling of the reviewer that the reader who is to take his first dip into abstract mathematics cannot very well do better than to read this elegant introduction to the logical foundations of algebra.

N. J. LENNES.

Leçons sur la Théorie de la Croissance. By ÉMILE BOREL. Paris, Gauthier-Villars, 1910. vi + 168 pp.

THIS book is one of the excellent series of monographs on the theory of functions appearing under the general editorial direction of M. Borel. It has grown out of two independent courses of lectures on the theory of increase (croissance) delivered by the author at the University of Paris during the winter semesters of 1907–1908 and 1908–1909 respectively. These two courses have been coordinated and unified by M. Arnaud Denjoy.

A function $f(x)$ is said to be increasing if $f(x') - f(x'')$ is positive when $x' - x''$ is positive. The theory of increase is devoted to the investigation of the rate of change of an increasing function $f(x)$ with respect to x as x approaches infinity.

The author begins his preface by giving expression to his growing conviction that the theory of increase is the essential