if the integral is taken in the ordinary or Riemannian sense, we must presuppose the integrability of the expressions $W(x)\cdot W_{n,r}(x)/[W_n(x)]^2$, but this is not necessary if the Lebesgue integral is used, provided, at least, the above expression is finite in $[a, b]$. The condition $W_n(x) \neq 0$ in $[a, b]$ may also be removed in certain cases, as when $W_n(x)$ vanishes at only a finite number of points, $x_0, x_1, \cdots, x_m$ in $[a, b]$ so that $\lim_{x=x_i} W_r(x)/W_n(x)$ exists and is finite for $\binom{r=0, 1, \ldots, n-1}{i=0, 1, \ldots, m}$. It seems unlikely, however, that formula (8) can be so extended as to give new criteria for linear dependence.

In conclusion we note that formulas (3), or (3′), and (8) taken together express G in terms of W in such a way as to show that $G=0$ when W vanishes, under the restrictions named. We thus have what may be regarded as the converse of (2). Similarly (8) and (2) express D in terms of G.

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NOTE ON INTEGRATION OF SERIES BY LEBESGUE INTEGRALS.

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LEBESGUE has shown in his work on integration that if a limited function, $f(x) \geqq 0$, is measurable for a measurable field A, it is "summable," or possesses a Lebesgue integral, and the value of this integral is the measure of the ordinate set Y, whose points are defined by the conditions: x in A, $0 \leqq y \leqq f(x)$. The converse is also true; that is, if Y is measurable, $f(x)$ is measurable and

$$\text{meas } Y = \int_A f(x)dA.$$

A proof of this may be found in Schoenfliess, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ergänzungsband II, part II, page 320.

It is the purpose of this note to show how by use of this idea the proof of Lebesgue's theorem on termwise integration of series can be greatly simplified and reduced to elementary theorems on point aggregates. The theorem in question is proved in Hobson's Theory of Functions of a Real Variable,

§§ 381, 382, 383. In the following proof the field A is taken as a one-dimensional set for clearness, but the proof holds for any number of dimensions.

THEOREM: Let $\{f_n(x)\}$ be a sequence of functions, integrable (in the sense of Lebesgue) and converging to $f(x)$ at all points of the measurable set A. Let the $f_n(x)$ be uniformly limited in A. Then $f(x)$ is integrable and

$$\int_A f(x)dA \equiv \lim_{n=\infty} \int_A f_n(x)dA.$$

Proof. Suppose first that each $f_n(x) \geqq 0$ in A. Let Y_n be the set of ordinates corresponding to $f_n(x)$. Then Y_n is measurable and

$$\text{meas } Y_n = \int_A f_n(x)dA.$$

Let U_n = union of $Y_n, Y_{n+1}, \cdots$. Then U_n is measurable and

$$(1) \qquad \text{meas } U_n \geqq \text{meas } Y_{n+p} \qquad (p \geqq 0).$$

Now let D = divisor $\{U_n\}$. Then D is measurable and

$$(2) \qquad \text{meas } D = \lim_{n=\infty} \text{meas } U_n \geqq \overline{\lim_{n=\infty}} \text{ meas } Y_n.$$

Evidently D consists of a set of ordinates. For let (x_1, y_1) be a point of D. Then (x_1, y_1) lies in every U_n and hence in an infinity of the Y_n. Hence the points (x_1, y), $y \leqq y_1$ lie in an infinity of the Y_n; hence in all the U_n and thus in D.

If any ordinate of D is not complete, add its upper limiting point and call the resulting set E. It is obvious that meas E = meas D. We now show that E is the ordinate set of $f(x)$ over A. Suppose the point (x_1, y_1) is the upper end of the ordinate of E through x_1. We must prove

$$(3) \qquad y_1 = y_2 = f(x_1).$$

Take $y = y_2 - \epsilon$. Since $f_n(x_1) \doteq f(x_1)$, for n_0 sufficiently great, the point (x_1, y) lies in all Y_n, $n \geqq n_0$; hence in all the U_n and in E.

$$(4) \qquad \therefore y_1 \geqq y = y_2 - \epsilon.$$

Now consider $y = y_2 + \epsilon$. Since $f_n(x_1) \doteq f(x_1)$, the point (x_1, y) lies in no Y_n for $n \geqq n_0$; hence not in all the U_n and thus not in E.

$$(5) \qquad \therefore y_1 \leqq y = y_2 + \epsilon.$$

Relations (4) and (5) give (3).

Hence E is the ordinate set of $f(x)$ and (2) gives

$$(6) \qquad \int_A f(x) dA = \text{meas } E \geqq \overline{\lim_{n=\infty}} \text{ meas } Y_n.$$

Now set $D_n =$ divisor $(Y_n, Y_{n+1}, \cdots)$ and $U =$ union $(D_1, D_2, \cdots)$. The sets D_n and U are measurable;

$$(7) \qquad \therefore \text{meas } U = \lim_{n=\infty} \text{meas } D_n \leqq \underline{\lim}_{n=\infty} \text{meas } Y_n.$$

As before, U is an ordinate set and addition of upper limiting points of ordinates gives V of the same measure. In a similar manner to the case of E, we can show that V is the ordinate set of $f(x)$ and get, with (7)

$$(8) \qquad \int_A f(x) dA = \text{meas } V \leqq \underline{\lim}_{n=\infty} \text{meas } Y_n.$$

Relations (6) and (8) give

$$(9) \qquad \int_A f(x) dA = \lim_{n=\infty} \text{meas } Y_n = \lim_{n=\infty} \int_A f_n(x) dA.$$

When the $f_n(x)$ are unrestricted in sign, we can set

$$\phi_n(x) = f_n(x) + C, \quad \text{where} \quad C \geqq \max_{n,\, x} |f_n(x)|,$$

and apply (9) to $\phi_n(x)$, giving

$$\begin{aligned} \lim_{n=\infty} \int_A f_n(x) dA &= \lim_{n=\infty} \int_A \phi_n(x) dA - \int_A C \cdot dA \\ &= \int_A \phi(x) dA - \int_A C \cdot dA \\ &= \int_A [\phi(x) - C] dA = \int_A f(x) dA. \end{aligned}$$

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