EISENHART'S DIFFERENTIAL GEOMETRY.

A Treatise on the Differential Geometry of Curves and Surfaces. By LUTHER PFAHLER EISENHART. Boston, Ginn and Company, 1909. xii + 474 pages.

THE plan of this book is similar to that of a number of mathematical treatises which have recently been published in the United States. It is intended to serve as an introduction to the subject of differential geometry for use in graduate courses; at the same time it contains accounts of recent researches on special topics by the author and by other writers which will be of interest to any reader and may well serve as an inspiration to further investigation on the part of the student.

The writer of this review has found the book an exceedingly useful one to have in the hands of students in a course in differential geometry. It contains numerous examples, some inserted in the text to illustrate particular discussions, and others of a more general character at the end of each chapter. Many of these examples are difficult for the elementary student to handle, but a suitable selection can readily be made. In the course referred to the book was used for reference, for the examples, and as a basis for reports by students. It was not found practical to follow closely the arrangement of material or the details of the discussions of particular subjects. These are matters, however, which depend much upon the individual taste of the instructor and the character of the students with whom he has to deal, and the fact that the book presented in many cases a different view-point from that of the lecturer seemed to enhance rather than to diminish its value.

In these days of the popularity and elegant methods of the theory of functions of a real variable, it is interesting to note that much of the theory of surfaces can be developed without the use of imaginaries, and to see also that the existence and uniqueness theorems for real differential equations can be applied with economy in many places. Professor Eisenhart has made use from the start of expansions in series. It is true of course in some cases, as for example with minimal curves and surfaces, that the theory would be incomplete or difficult to phrase without the introduction of analytic functions. But it is certainly doubtful in many others if any simplification is effected by their use.

One of the most interesting features of the book is the introduction of movable axes, the so-called "moving trihedrals," for twisted curves as well as surfaces. The fundamental rotations and translations associated with these trihedrals are freely applied in the writings of Darboux and others, and their applications rival in importance those of the Gaussian fundamental quantities for a surface. The student will hardly find himself at home in the literature of to-day without an intelligent acquaintance with them. Professor Eisenhart has developed the properties of these quantities without the help of the notions from mechanics used by Darboux, and has applied them in so many situations that their importance and uses seem perfectly clear to the reader.

The book would be somewhat more convenient if the chapter and section numbers had been printed at the top of each page, as there are numerous references to sections and to numbered equations in the different chapters which often require considerable turning. The typography is otherwise excellent, and a good index adds much to the usefulness of the work.

The contents of the book may be roughly divided into four parts. The first six chapters contain the fundamental principles of the theory of twisted curves and surfaces, and these are followed in Chapters VII and VIII by applications to a variety of special cases, including quadrics, ruled surfaces, minimal surfaces, surfaces of constant curvature, and surfaces with plane or spherical lines of curvature. One of the topics most fully treated is the theory of the deformation of surfaces in Chapters IX–XI. Chapters XII on rectilinear congruences, XIII on congruences of circles orthogonal to a family of surfaces, and XIV on triply orthogonal systems of surfaces form the concluding group and with Chapter VIII furnish most interesting reading.

The theory of twisted curves in space is developed in Chapter I. The osculating plane is defined as the limiting position of a plane through a tangent and a neighboring point M of the curve as M approaches the point of tangency, and the osculating sphere is defined as a sphere with its center in a proper position on the polar axis. It seems in some ways more desirable to define these as limiting positions of a plane or sphere through three or four neighboring points of the curve. The same methods may then be applied to both of them as well as to the determination of the tangent line and osculating circle, and the method itself is important in other connections.

One is impressed here at the start with the importance of the existence and uniqueness theorems for a system of differential equations. In discussing the determination of a curve by means of its intrinsic equations, as in many other places in the book, the author has such a system to consider. It would certainly mean economy of thought for the reader, and perhaps even economy in printed pages, if these theorems were stated somewhere in a form suited to the applications which are made of them. With them in mind the intrinsic equation theory could be quite concisely stated.

The sections devoted to the moving trihedral and its applications are interesting ones. The notions involved seem difficult for the elementary student to handle, but they are exceedingly important. Professor Eisenhart has derived the fundamental equations

$$\frac{\delta\xi}{ds}=\frac{d\xi}{ds}+1-\frac{\eta}{\rho}, \qquad \frac{\delta\eta}{ds}=\frac{d\eta}{ds}+\frac{\xi}{\rho}+\frac{\zeta}{\tau}, \qquad \frac{\delta\zeta}{ds}=\frac{d\zeta}{ds}-\frac{\eta}{\tau}$$

by means of expansions in series. The quotients in the first members of these equations determine the direction of the absolute motion in space of the point whose coordinates are ξ, η, ζ with respect to the moving axes, while those in the second members give the motion relative to the moving axes themselves. If the coordinates ξ', η', ζ' with respect to a fixed trihedral at a point M' of the curve are expressed in terms of the coordinates ξ, η, ζ with respect to a trihedral with vertex at a neighboring point M, the formulas just given can be very simply derived by differentiation and use of the initial values given by Professor Eisenhart on page 30. The use of symbols for differentiation in the left members of the equations above is unfortunate. The direction $\delta\xi/ds : \delta\eta/ds : \delta\zeta/ds$ cannot be found by differentiation of ξ, η, ζ and it would be better to denote it by other symbols, say $V_\xi : V_\eta : V_\zeta$. A number of applications of the formulas above have been made in the text. The properties of involutes and evolutes of a curve can also be derived in this way, as is suggested in example 8, page 47, apparently more simply than by the methods used in § 21.

This chapter contains also a short discussion of the theory ot the integration of a Riccati equation, a study of developable surfaces as the tangent surfaces of curves, and a section on minimal curves.

The more elementary portions of the surface theory are presented in Chapters II–IV. In Chapter II the notion of curvilinear coordinates is introduced, and the theory of envelopes with applications to developable surfaces and envelopes of families of spheres is explained. The discussion of the determination of characteristic curves in terms of the moving trihedral would be clearer to the reviewer if the method suggested above for the derivation of the fundamental equations in Chapter I had been used.

In Chapter III the quantities E, F, G are defined and those parts of the surface theory which depend upon them exclusively are developed. It is to be regretted that a precise definition of a differential parameter was not set down. Roughly expressed, a differential parameter is a function of E, F, G, ϕ, ψ, $\cdots$ and their derivatives with respect to u, v, which remains unchanged in form when these functions are replaced by their equivalents in terms of E_1, F_1, G_1 and derivatives with respect to two new variables u_1, v_1. According to this definition the functions E, F, G are not themselves invariants, as is stated at the bottom of page 87. And it also does not seem to be true that a function expressible in terms of differential parameters with respect to one system of coordinates will be equal to the same expression formed with respect to another, unless the function is itself invariant. The proof given on page 165 for the invariance of $\Delta_{22}\theta$ would therefore seem to be only a proof of formula (38) for expressing $\Delta_{22}\theta$, known by other means to be invariant, in terms of other differential parameters.

In Chapter IV the second fundamental quantities D, D', D'' appear for the first time, and the well known relations between the curvatures of lines through a point on a surface are set down. The reader is also introduced to lines of curvature and conjugate systems, asymptotic lines, geodesics, and the spherical representation of a surface. These conceptions appear again in Chapter VI for a more detailed discussion with the help of the Christoffel symbols, the moving trihedral, and the fundamental equations derived in Chapter V. On page 117 the formula near the bottom of the page should read

$$\frac{d^2x}{ds^2} = \frac{\partial^2 x}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2 x}{\partial u \partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2 x}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial x}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial x}{\partial v}\frac{d^2v}{ds^2},$$

though the argument which follows is unaltered by the omission of the last two terms. On page 123 the statement that a surface lies on only one side of the tangent plane at points of parabolic curvature is evidently incorrect. This is the old question of the minimum of a function whose expansion begins with a quadratic form having a vanishing discriminant. One needs only to consider the surface generated by revolving a sine curve about a tangent at one of its maximum points. The lines along which the curvature is parabolic are generated by the points of inflexion.

In the next chapter the equation of Gauss expressing the curvature of a surface in terms of $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ and their derivatives is derived, as well as the two so-called Codazzi equations, and a proof is given of the fundamental theorem of surface theory. This is the theorem which states that any six functions E, F, G, D, D', D'' which satisfy the Gauss and Codazzi equations are the fundamental quantities of a surface which is uniquely defined except for its position in space. Other forms of the theorem are also given in which the surface is characterized by D, D', D'' and the functions $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ defining its spherical representation, or by the direction cosines X, Y, Z of its normal together with the distance W of the tangent plane from the origin. The functions X, Y, Z, W are called the tangential coordinates of the surface.

A surface may also be characterized by functions of u, v representing the rotations and translations of a system of axes which move so that the xy-plane is always tangent to the surface at the origin, and in such a way that the x-axis always makes an angle $U(u, v)$ with the curves $v =$ constant. This system of axes is called the moving trihedral of the surface. Professor Eisenhart has derived the equations of condition which must be satisfied by the six rotations p, q, r, p_1, q_1, r_1 and the four translations ξ, η, ξ_1, η_1 of the moving trihedral with the help of the moving trihedral for a curve already discussed in Chapter I, and his method seems simpler than that of Darboux. Any ten functions $p, q, r, \cdots, \eta$ which satisfy these equations define a moving trihedral enveloping a surface which is uniquely determined except for its position in space. The latter part of the chapter is devoted to the determination of the principal properties of a surface in terms of the rotations and translations above mentioned, and to the application of these results in the cases of parallel surfaces and evolutes of a given surface.

With the formulas of the preceding chapter at hand it is possible for the author, in Chapter VI, to study more comprehensively the theory of systems of curves on a surface. In the first half of the chapter the conditions are derived which must be satisfied by functions E, F, G in order that the parametric lines may be asymptotic, or by $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ in order that the parametric lines on the sphere may represent a system of asymptotic or isothermal conjugate lines on the surface, and in each case it is shown how the surfaces themselves may be determined. A number of theorems are also given showing how surfaces with asymptotic or conjugate parametric lines are determined by the solutions of certain partial differential equations of the second order. The last half of the chapter is devoted to geodesics and geodesic coordinate systems. A proof is given of the celebrated theorem of Gauss concerning the sum of the angles of a geodesic triangle, and the minimizing properties of geodesics are briefly discussed from the standpoint of the calculus of variations.

The next two chapters contain applications of the theory of surfaces to special cases. The properties of systems of lines of various kinds on quadrics and ruled surfaces is developed, and an interesting discussion of the more important properties of minimal surfaces is given. In Chapter VIII surfaces of constant curvature are treated in the first part. There are four kinds of transformations of such surfaces, called the transformations of Hazzidakis, Bianchi, Bäcklund, and Lie. The result of these transformations is the determination of surfaces of constant curvature having the same curvature as that of the given one and related to it by some geometrical or analytical process. Thus for the transformation of Bianchi it is found that if circles of radius a are described in the tangent planes to a surface of constant curvature $-1/a^2$, with centers at the points of contact, then these circles are the orthogonal trajectories of an infinity of surfaces of curvature $-1/a^2$. The transformation of Bäcklund is a generalization of this transformation, while those of Hazzidakis and Lie are analytical in character. It may be of assistance to the reader to remark that §§ 118–119 are much clearer if reference is made to § 77, page 190, and § 82, pages 199, 200. In fact the argument does not seem to be complete without these references, though no mention is made of them. W-surfaces are surfaces for which a relation of some sort exists between the principal radii of curvature. They include minimal surfaces and surfaces of constant mean curvature. After discussing them the author devotes the last part of Chapter VIII to surfaces which have systems of lines of curvature of special kinds. Examples of these are the surfaces of Monge which have one system of plane geodesic lines of curvature, and which are generated by a plane curve whose plane rolls without slipping over a developable surface. The equations at the bottom of page 313 seem to be misprinted. They should read

$$\xi = \frac{u}{2(1+u^2)^{\frac{1}{2}}}\left[u^2 + \tfrac{1}{2} + a - 2(1+u^2)^{\frac{1}{2}}\right],$$

$$\zeta = \frac{1}{2(1+u^2)^{\frac{1}{2}}}\left[u^2(u^2 + \tfrac{1}{2} + a) - (u^2 - \tfrac{1}{2})(1+u^2)^{\frac{1}{2}}\right],$$

$$\rho = \tfrac{1}{2}(u^2 + \tfrac{1}{2} + a)(1 - u^2)^{\frac{1}{2}}.$$

The chapters in which the deformation of surfaces is discussed have been mentioned above. In determining all the surfaces with a given linear element

$$E du^2 + 2F du\, dv + G dv^2$$

by the method of Darboux, it is found that whenever a solution $z(u, v)$ of the equation

$$\Delta_{22} z = (1 - \Delta_1 z) K$$

is known, where the differential parameters and curvature K are formed with respect to the quadratic form just given, then two other functions $x(u, v)$, $y(u, v)$ can be determined by quadratures, which with $z(u, v)$ define a surface having the length element required. The method of Weingarten, published in a memoir which was awarded the grand prize of the French Academy in 1894, makes use of the moving trihedral. A set of coordinates on the surface can be so chosen that

$$r = v, \quad r_1 = 0, \quad K\sqrt{EG - F^2} = 1,$$

where r and r_1 are two of the rotations of the trihedral. The equations which must be satisfied by the translations of the trihedral are then

$$\frac{\partial \xi}{\partial v} - \frac{\partial \xi_1}{\partial u} = -v\eta_1, \quad \frac{\partial \eta}{\partial v} - \frac{\partial \eta_1}{\partial u} = v\xi_1,$$

$$p\eta_1 - \eta p_1 + \xi q_1 - q\xi_1 = 0.$$

The last equation can be expressed entirely in terms of u and differential parameters of u with respect to the length element of the spherical representation of the x-axis of the moving trihedral. When this is done, any other solution $u_1(u, v)$ of the differential equation so found furnishes a surface applicable to the original one. In the latter part of the chapter the results of Weingarten are given a different form and applied to the determination of surfaces applicable to surfaces of revolution and other special cases. It is not clear from the text that the functions ξ, η, ξ_1, η_1 can be chosen in the form (29) of page 357 for any surface, as is implied in the theorems of page 360.

If a family of surfaces of the form

$$x' = x(u, v) + \epsilon x_1(u, v) + \epsilon^2 x_2(u, v) + \cdots,$$
$$y' = y(u, v) + \epsilon y_1(u, v) + \epsilon^2 y_2(u, v) + \cdots,$$
$$z' = z(u, v) + \epsilon z_1(u, v) + \epsilon^2 z_2(u, v) + \cdots,$$

were such that each had the same length element, then any one of them would be applicable to every other. The problem of the so-called infinitesimal deformation of the surface $S(\epsilon = 0)$, is the determination of the first coefficients x_1, y_1, z_1 in such an expansion. It turns out that the problem is equivalent to the determination of a surface S_1 corresponding to S in such a way that corresponding curves on the two are orthogonal, and this depends upon the solution of a partial differential equation of the second order. The latter problem is again equivalent to the determination of a surface S_0 associate to S, i. e., one such that tangent planes at corresponding points of S and S_0 are parallel, and such that asymptotic lines on either surface correspond to a conjugate system on the other. These results are derived in Chapter XI with further interrelations between the triples of surfaces S, S_0, S_1.

One of the most interesting applications of differential geometry, on account both of the elegance of the theorems and also of the opportunities for the exercise of geometric imagination which present themselves, is the theory of congruences, to which Chapter XII is devoted. Through any one of a double infinity of straight lines there pass an infinity of ruled surfaces belonging to the congruence, whose lines of striction all cut the given line on a certain segment. The end points of the segment correspond in this way to two principal ruled surfaces whose tangent planes at the end-points are perpendicular to each other. Similarly through each line there pass two developable surfaces whose cuspidal edges are tangent to the line at two focal points equidistant from the end-points just mentioned. The totality of the focal points on all the lines of the congruence form two focal surfaces. After developing these conceptions and their interrelations with each other, the author takes up the study of special congruences characterized for the most part by restrictions of various kinds placed upon their focal surfaces. Thus pseudospherical congruences have pseudospherical focal surfaces, and congruences of Guichard are those whose developables touch the focal surfaces along lines of curvature.

A cyclic system is a congruence of circles which has a single infinity of orthogonal surfaces. With every such system is associated a congruence of lines, the axes of the circles, called a cyclic congruence. Conversely, with every cyclic congruence there is associated but one cyclic system, unless the congruence is of the special type called congruences of Ribaucour. In § 178 Professor Eisenhart has exhibited a transformation of surfaces which have the same spherical representation of lines of curvature as pseudospherical surfaces, into others of the same type. This is a generalization due to the author of the transformation of Bianchi for pseudospherical surfaces.

The concluding chapter of the book is devoted to triply orthogonal systems of surfaces, examples of which are provided by the orthogonal surfaces of a cyclic system, taken with the two families of surfaces generated by the circles of the system which intersect them in their lines of curvature. The determination of triply orthogonal systems which have one family consisting of surfaces of revolution or of pseudospherical surfaces, is considered, and it is shown, following Darboux, that the general problem of the determination of all triply orthogonal systems may be reduced to the solution of a partial differential equation of the third order.

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