STUYVAERT'S STUDIES IN ANALYTIC GEOMETRY.

Cinq Études de Géométrie analytique. Par M. STUYVAERT. Gand, E. Van Goethem, 1908. vi + 230 pp.

THIS book, consisting of applications of the theory of matrices and elimination, was awarded the "Prix François Deruyts" by the Royal Academy of Belgium and forms an important addition to the methods of analysis used in geometry. Whether it is a question of elimination or superelimination (the study of the conditions that two equations have more than one root in common), the result leads to the vanishing of a rectangular matrix, i. e., to the existence of a certain linear relation or "faisceau" of such between the elements of each row or column, and from the details of structure of this matrix can be read the properties of the variety represented by equating it to 0. The results are in general known, so the author seldom trusts the reader to discover the new theorems, but instead labels them as such. The originality and value of the book consist in the methods used, and it is surprising with what simplicity and elegance the machinery works.

The first of the five studies of which the book consists ("Applications géométriques de la théorie des matrices") investigates those processes of elimination which give rise to matrices and the use which can be made of this theory for the determination of special elements of geometrical figures. The conditions that two equations in x have at least two common roots can be expressed by the vanishing of a matrix in l lines and $l+1$ columns. If the elements of such a table are ternary or quaternary forms, we have respectively the representation of a finite number of points or a skew curve. Such a representation of a curve gives at once, by means of tables obtained from the original matrix by the suppression or adjunction of various lines and columns, its order, genus, modes of generation, circumscribing surfaces, multisecant curves, groups of remarkable points, and other properties.

After deriving general formulas for the orders and genus of curves represented by vanishing matrices the author discusses skew curves of orders three to ten. The following will illustrate his methods: If a_x, $\cdots$ represent linear quaternary forms, the relations

$$\left\|\begin{matrix} a_x & b_x & c_x \\ a'_x & b'_x & c'_x \end{matrix}\right\| = 0$$

represent the most general skew cubic. Its system of bisecants is $\lambda a_x + \mu b_x + \nu c_x = 0$, $\lambda a'_x + \mu b'_x + \nu c'_x = 0$. If we precede the above matrix by a line of constants α, β, γ, the resulting equation gives ∞^2 quadrics circumscribing the curve. Similarly

$$\left\|\begin{matrix} a_x & b_x & c_x \\ a'_x & b'_x & c'_x \\ a''_x & b''_x & c''_x \end{matrix}\right\| = 0$$

is a skew C_6 of genus 3. As the four equations of the type $\lambda a_x + \lambda' a'_x + \lambda'' a''_x = 0$ are compatible in λ for all points of C_6, the latter is the locus of intersections of homologous planes of four collinear "gerbes." In a similar way we can see that it is the locus of points of intersection of homologous rays of three superimposed projective spaces. If we precede the matrix of C_6 by a column of constants λ, the resulting table annuls itself for four points of C_6. These four points annul an evident determinant containing eight arbitrary constants whose vanishing gives a quadric, which is seen to be circumscribed to a variable skew cubic cutting C_6 eight times, and whose equation comes from the above determinant. Thus if we vary λ, λ', λ'', we have on C_6, ∞^2 quadruples of points such that each of these groups can be joined by a quadric to all skew cubics cutting C_6 eight times.

When matrices of l lines and $l + 1$ columns of independent linear forms are used, the resulting curve is of order $\frac{1}{2}l(l + 1)$; hence in order to discuss skew curves of order 5, 7, 8 and 9 the author introduces non-linear forms, e. g., C_5 is

$$\left\|\begin{matrix} a_x^2 & b_x & c_x \\ a'^2_x & b'_x & c'_x \end{matrix}\right\| = 0.$$

The fact that non-linear forms had to be introduced suggests one of the most interesting problems for research in this subject namely: "The determination of what curves are representable by matrices."

The section on the Jacobian of a system of three or five surfaces, i. e., the skew curve defined by the vanishing of the usual Jacobian matrix is of value from the standpoint of results, as the author's theorems are more complete than those of Cremona or Salmon. The problem of geometrical loci, i. e., the problem of eliminating t between $F(x, y, z, t) = 0$ and $f(x, y, z, t) = 0$, where the common roots in t may have various multiplicities, is well treated and general results are derived. The application of these results as a corollary to the very important subject of envelopes is however disappointing, being confined to an example. The study closes with sections on the multisecants of rational curves, the multisecant conics of skew curves, the elimination of two or more unknowns from a system of equations, and applications to space of four dimensions. The latter suggests another line for investigation, for when the number of homogeneous variables surpasses four, the algebraic problems are greatly changed, so that for example it is necessary to study matrices the difference of the numbers of whose columns and lines is greater than two, and also matrices whose elements are dependent forms.

In the first study the author limits himself to matrices whose elements are forms in a single variable. The case where they are forms of two or more series of variables and its application to particular cases furnishes the basis of a theory of congruences of skew curves, a theory which is also still in the making and is the subject of Study II: "Congruences de variétés algébriques annulant des matrices." The author considers the matrix of l lines and $l + 1$ columns containing homogeneous forms in $x_1, \cdots, x_d$, and $\alpha_1, \alpha_2, \alpha_3$. Its vanishing represents a congruence of varieties of $d - 3$ dimensions in a space of $d - 1$ dimensions, and the writer seeks under the easiest hypotheses how α must enter in order that an arbitrary system of values of x belong to a single variety of the congruence. After general theorems he limits himself to matrices of two lines and three columns of forms linear in both x and α, and classifies the linear congruences so obtained into six types. If $d = 4$ we have a theory of linear congruences of skew cubics and if $d = 3$ of linear congruences of coplanar triangles.

The author defines a singular point as one belonging to a simple infinity of members of a congruence and discusses the singular points in the above types and the figures they represent, cleverly reducing this problem to a study of the loci represented by the formulas distinguishing the different types of congruences; for example, if $d = 4$, we have for type III, defined by

$$\left\| \begin{matrix} \alpha_1 a_x + \alpha_2 b_x & \alpha_1 a'_x + \alpha_2 b'_x & \alpha_1 a''_x + \alpha_2 b''_x \\ \alpha_1 d_x + \alpha_3 g_x & \alpha_1 d'_x + \alpha_3 g'_x & \alpha_1 d''_x + \alpha_3 g''_x \end{matrix} \right\| = 0,$$

a congruence of skew cubics having for directrices a sextic of genus 3 and two skew cubics, each curve of the congruence and the two cubic directrices cutting the sextic eight times and each curve cutting the two cubic directrices once.

The third study, "La théorie des matrices dans l'espace réglé" is very short but suggestive of what more can be done. It contains sections on matrices with l lines and $l + 3$ columns (whose vanishing as soon as the number of variables surpasses four is in general compatible), congruences of straight lines annuling matrices, vanishing of the first minors of a determinant, and complexes annuling a determinant. The matrix M of l lines and $l + 1$ columns whose elements are homogeneous forms of plückerian line coordinates when set equal to zero represents a congruence Γ of straight lines. If we precede M by a line of constants α, we get a complex C_α including Γ. Two such complexes C_α and C_β have Γ and a congruence Δ in common, the latter annuling the matrix obtained by preceding M with the lines α and β. Δ and Γ have a ruled surface in common.

The finding of the points which annul all the first minors of a determinant (a_{ik}) $(i, k = 1, 2, 3)$ is reduced to finding the points common to two vanishing matrices which represent surfaces. If we have variables $x_1, \cdots, x_d$ the vanishing of all the first minors of a determinant of nine elements gives in space of $d - 1$ dimensions a variety of $d - 5$ dimensions. If in the above determinant the variables are again line coordinates, its vanishing gives a complex C whose double elements annul all the determinant's first minors and belong to ∞^1 ruled surfaces. If the determinant is preceded by a line of constants we have a congruence and so again get an interesting relation between complexes, congruences, and ruled surfaces.

Study IV, "Sur une forme doublement quadratique binaire et symétrique" serves as an introduction to the last article and concerns a form which in non-homogeneous variables is

$$\phi = a - x^2y^2 + bxy + c + h(xy)(x + y) + g(x^2 + y^2) + f(x + y).$$

In this form x and y stand for abscissas of two "ponctuelles" and not cartesian coordinates. ϕ is used for a classification of symbolic forms and finally applied to conics and surfaces, where the author deals chiefly with groups of elements harmonically situated, such treatment having been suggested by Halphen who in his Traité des fonctions elliptiques reduced the study of two coplanar conics to that of a form similar to ϕ, whose geometric meaning was that it set up the most general correspondence on a conic by means of tangents to another conic. The study closes with ruled surfaces having for generators the bisecants of a skew cubic and as a special case an interesting surface of the fourth order defined, for example, as the envelope of quadrics determined by couples of homologous elements of two projective series of bisectants.

If ϕ ceases to be symmetric and x, y are coordinates in a plane, the author has a basis for his last study "Quadrilatères de Steiner dans certaines courbes et surfaces algébriques" which is remarkable in the beauty and symmetry of its analysis and theorems. The author first proves the celebrated theorem of Steiner, "A plane binodal quartic is not in general circumscribed to a quadrangle having two of its diagonal points at the nodes; but if there exists one such quadrangle there is an infinity, in which case the third diagonal point describes a conic and the sides of the quadrangle which do not pass through the nodes envelope a curve of the fourth class," and calls such quadrangles "quadrillée." The necessary and sufficient conditions that a quartic be quadrillée are derived as well as many theorems on quadrillée quartics of two, three and four nodes.

The author next seeks, by original methods, the properties to the skew quartic of the first kind C_4' which correspond of those explained for plane quartics, using in his analysis quadrillée quadrics (quadrics of Voss) and quadrillée cones, i. e., perspective cones of C_4' every plane trace of which is a quadrillée quartic. We shall note one theorem as it is proven without the customary use of elliptic functions. "Through every skew C_4' there pass six quadrillée quadrics. In the system $F_4 + kS_4$ having C_4' as a base there are four cones defined by a biquadratic form in k and the six quadrillée quadrics annul the sextic covariant of this form." This section is followed by one on special surfaces of the fourth order having a simple infinity of double points. These double points can be divided in general into ∞^2 couples and through each of these couples pass six quadrillée sections whose ∞^2 planes envelope a surface. The ∞^3 diagonals of the quadrangles generate a complex and the third diagonal points are on ∞^2 conics. The book closes with some paragraphs on graphical tables for the representation of functions, where the author solves the problem which when proposed to him suggested this last study, namely, to determine when the equation of a certain surface of the sixth order could be put in the form of a determinant equated to zero.

The very power of Professor Stuyvaert's methods, which cover every page with italicized theorems, makes the book monotonous to read, but at the same time gives it value as a compendium. As we have pointed out throughout this review, the author of these five studies has made a large addition to the instruments of a modern geometer and at the same time pointed out the way to other fields of investigation, perhaps the best of which is to discover how to do away with the limitations of the above methods in which to a large extent the curve has to be found to fit the formulas.

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SHORTER NOTICES.

Études sur Léonard de Vinci, ceux qu'il a lus et ceux qui l'ont lu. Par PIERRE DUHEM. Paris, A. Hermann, 1906, 1909. 8vo. 2 volumes. x + 355, iv + 469 pp. 27 francs.

IT would be difficult to find a man better fitted for the preparation of a work of this character, either by temperament or because of scholastic attainments, than the eminent professor of theoretical physics of the Faculty of Sciences at Bordeaux. Writing of a great genius whose wide range of information covered all that was then known of physics, of a man whose interest in France showed itself in divers activities, of a scholar who was in correspondence with the learned men of his time, and of an inventor whose fertile imagination was seen in a large number of helpful devices, M. Duhem had a most inspiring subject and one which he could hardly have failed to treat *con amore.*

The author confesses at the beginning to the feeling of awe that such a topic would naturally create in any serious mind that contemplates the work of this master. In his own case, he tells us, this was followed by a period of contemplation, more free from emotion, in which the idea was prominent that