The cases in which the Jacobian determinant of $T$ has at least one non-zero element, say $f_0(0, 0) \neq 0$, are completely discussed. Certain cases where all $f_0, f_0, \phi_0, \phi_0$ are zero when $u = 0, v = 0$ are treated. If $f$ and $\phi$ admit a common factor in $R$, then there is an explosive point in $\mathbb{R}$, having an infinitely many valued inverse. Even then $\mathbb{R}$ may be the complete neighborhood of this point, the number of branches which are continuous outside this point being different in different sub-regions of $\mathbb{R}$.

53. It is well known that the group of isomorphisms of a group of order $p$ is of order $p - 1$, and that of a cyclic group of order $p^2$ is of order $p(p - 1)$. The corresponding group of the non-cyclic group of order $p^2$ is simply isomorphic with the linear homogeneous group on $p^2$ variables.

The groups of isomorphisms of all types of groups of order $p^3$ are determined by Western in his paper on "Groups of order $p^3$," Proceedings of the London Mathematical Society, volume 30.

Professor Marriott has determined the groups of isomorphisms of all types of groups of order $p^4$. He exhibits these as substitution groups and determines the order of each.

F. N. Cole,
Secretary.

ON THE NEGATIVE DISCRIMINANTS FOR WHICH THERE IS A SINGLE CLASS OF POSITIVE PRIMITIVE BINARY QUADRATIC FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, April 29, 1911.)

For such a discriminant — $P$, the problem of the representation of numbers by a binary quadratic form of discriminant — $P$ is quite elementary; moreover, factorization into primes is unique in a quadratic field of discriminant — $P$. The only *

* E. Landau, Mathematische Annalen, vol. 56 (1903), p. 671. His method is not applicable to discriminants — $P$, where $P$ is odd, as was pointed out by M. Lerch, ibid., vol. 57 (1903), p. 568. Results obtained by the latter by use of a relation between numbers of classes will here be proved by more elementary means and extensions given.
such discriminants of the form \(-4k\) are those having \(k = 1, 2, 3, 4, 7\), as was conjectured by Gauss after an examination of the determinants as far as \(-3000\). The present note gives practical criteria and the result of an examination of the values of \(P\) less than one and one half million. We denote \(ax^2 + bxy + cy^2\) by \((a, b, c)\) and call \(b^2 - 4ac\) its discriminant.

First, let \(P \equiv 0 \pmod{4}\). Then \((1, 0, \frac{P}{4})\) must be the only reduced primitive form of discriminant \(-P\). The case in which \(P/4\) is divisible by two distinct primes is excluded, since we may then express \(P/4\) as the product of two relatively prime factors \(a, c\), such that \(1 < a < c\), and hence obtain the new primitive reduced form \((a, 0, c)\) of discriminant \(-P\). Hence \(P = 4p^e\), where \(p\) is a prime. For \(p = 2\), \((4, 4, 2^{e-3}+1)\) is a primitive reduced form of discriminant \(-P\) if \(e \geq 4\), and \((3, 2, 3)\) is one if \(e = 3\); while for \(e = 1\) or 2, whence \(P = 8\) or 16, there is a single primitive reduced form. Next, let \(p > 2\). The even number \(p^e + 1\) cannot have an odd factor > 1, since otherwise it would equal the product of two relatively prime integers \(a\) and \(c\), such that \(1 < a < c\), and \((a, 2, c)\) would give a new primitive reduced form of discriminant \(-P\). Hence \(p^e + 1 = 2^k\). Then \((8, 6, 2^{k-3}+1)\) or \((5, 4, 7)\) is a primitive reduced form of discriminant \(-P\) if \(k > 5\) or \(k = 5\), respectively. For \(k = 4\), \(2^k - 1 = 15\) is not a power of a prime. For \(k = 1, 2, 3\), \(P = 4, 12, 28\), there is a single primitive reduced form.

Next, let \(P \equiv 3 \pmod{4}\). Then \([1, 1, \frac{1}{4}(1 + P)]\) must be the only reduced primitive form of discriminant \(-P\). If \(P = rs\), where \(r\) and \(s\) are relatively prime and \(> 1\), one of the factors is \(\equiv 3 \pmod{4}\) and the other \(\equiv 1 \pmod{4}\). Let \(r > s\). Then \([(r + s)/4, (r - s)/2, (r + s)/4]\) is a new primitive form of discriminant \(-P\), which is reduced if \(3s \leq r\). Its second right neighboring form (obtained by using \(\delta = -1, \delta' = 0\)) is \([s, -s, \frac{1}{4}(r + s)]\), which is reduced if \(3s < r\). Hence \(P = p^e\), where \(p\) is a prime \(\equiv 3 \pmod{4}\) and \(e\) is odd. If \(p > 3, e \geq 3\), the form with \(a = \frac{1}{4}(p + 1), b = 1, c = (p^e + 1)/(p + 1)\) is a new primitive reduced form of discriminant \(-P\); indeed, \(e > 4a\) since \(p^e - 1 \geq p^2 > p + 2\). For \(P = 27\), \((1, 1, 7)\) is the only primitive reduced form. For \(P = 3^3\), \([9, 3, \frac{1}{4}(3^{e-2} + 1)]\) or \((7, 3, 9)\) is a primitive reduced form if \(e > 5\) or \(e = 5\), respectively. Thus, if \(P \neq 27, P\) must be a prime. Set

\[T_j = \frac{1}{4}[(2j + 1)^2 + P] = T_0 + j(j + 1).\]

* Disquisitiones Arithmeticae, Art. 303.
If \( j = qm + r \), \( 0 \leq r < m \), then \( T_j \equiv T_r \equiv T_{r-1} \pmod{m} \). For \( r > \frac{1}{3}(m-1) \), \( m-r-1 < \frac{1}{3}(m-1) \). Hence any \( T_j \) is congruent modulo \( m \) to some \( T_r \), where \( 0 \leq r < \frac{1}{3}(m-1) \). Let \( 2g + 1 \) be the greatest odd integer \( \leq \sqrt{P/3} \). In a reduced form \( (a, b, c) \), \( b > 0 \), we have \( b = 2\beta + 1 \equiv 2g + 1, \beta \equiv g. \) We shall prove that there is a single reduced form of discriminant \( -P \) if and only if \( T_0, T_1, \ldots, T_{\beta-1} \) are all prime numbers. If they are primes, a reduced form has \( a = 1, b = 1 \). Conversely, let there be a single reduced form. If \( T_0 \) were composite, there would be a reduced form with \( b = 1, a > 1 \). Suppose that \( T_0, \ldots, T_{\beta-1} \) are primes, but \( T_\beta = ac \), \( c \equiv a > 1 \), where \( 0 < \beta \leq g \). If \( a \equiv b \), where \( b = 2\beta + 1, (a, b, c) \) would be reduced. Hence \( a < b \). Applying the above result for \( m = a \), we see that \( T_\beta \equiv T_r \pmod{a} \), where \( r \) is some integer \( 0 \leq r < \frac{1}{3}(a-1) \). Thus \( r < \beta \), so that \( T_\beta \) is a prime. But \( T_\beta \equiv T_\beta \equiv 0 \pmod{a} \). Hence \( T_\beta = a \). Thus \( a \equiv T_\beta \equiv \frac{1}{2}(1 + P) \). \( P \equiv 3(2g + 1)^2 \equiv 3(2\beta + 1)^2 > 3a^2, a > \frac{1}{2}(1 + 3a^2) \). Thus \((3a - 1)(a - 1) < 0 \), which contradicts \( a > 1 \).

If \( P \) is a prime \( < 27 \), then \( g = 0 \) and the condition is that \( T_0 = \frac{1}{2}(1 + P) \) shall be a prime. This is satisfied when \( P = 3, 7, 11, 19 \).

For \( P \equiv 7 \pmod{8} \), \( P > 7 \), \( T_1 \) is even and \( > 2 \).

For \( P \equiv 3 \pmod{8} \), set \( P = 8k - 5 \). For \( k \equiv 2 \pmod{3} \), \( k \geq 5 \), \( T_0 = 2k - 1 \) is divisible by 3 and exceeds 3; while for \( k = 2 \), \( P = 11 \). For \( k \equiv 1 \pmod{3} \), \( P \) is divisible by 3. For \( k \equiv 0 \pmod{3} \), \( P = 24t - 5 \). For \( t = 1, 4, \) or 0 (mod 5), \( T_0 = 6t - 1, T_1 = 6t + 1 \) or \( P \) is divisible by 5 and exceeds 5 except when \( t = 1, P = 19 \). For \( t = 2 \) or 3, \( P = 43 \) or 67 and \( g = 1 \), while \( T_0 \) and \( T_1 \) are primes. For \( t = 7, P = 163 \), \( g = 3 \), and \( T_0, T_1, T_2, T_3 \) are primes \( 41, 43, 47, 53 \). For \( t = 8 \), \( P = 11 \cdot 17 \). There remain the cases \( t = 5l + 12, 5l + 13 \), where \( l \geq 0 \). Hence we may state the

Theorem. There is a single class of positive primitive quadratic forms of negative discriminant \( -P \) when

\[ P = 3, 4, 7, 8, 11, 12, 16, 19, 27, 28, 43, 67, 163; \]

but more than one class if \( P \) is not one of these 13 numbers and not a prime of the form \( 120l + 283 \) or \( 120l + 307, l \geq 0 \).

The remaining primes \( < 1000 \) are \( P = 283, 523, 643, 883, 307, 547, 787, 907 \). For these \( g \geq 4 \), \( T_2 = 77, T_1 = 7 \cdot 19, T_0 = 7 \cdot 23, T_0 = 13 \cdot 17, T_0 = 77, T_2 = 11 \cdot 13, T_2 = 7 \cdot 29, \)
\[ T_1 = 13 \cdot 19, \] respectively. Hence in each case there is more than one class.

A practical method of examining a wide range of values of \( P \) consists in first excluding the values of \( l \) for which any one of the numbers \( T_0, \ldots, T_6 \) has a given small prime factor \( p \). For \( P = 120l + 283 \) or \( 307 \), \( T_0 = 30l + 71 \) or \( 30l + 77 \), \( g \equiv 4 \). This exclusion has already been effected for \( p = 3 \) or 5.

For \( p = 7 \), any \( T_i \) is congruent to \( T_0, T_1, T_2 \) or \( T_3 \). For \( T_0 \equiv 30l + 77 \), these are divisible by 7 if \( l \equiv 0, 6, 4, 1 \) (mod 7), respectively; for \( 30l + 71 \), if \( l \equiv 3, 2, 0, 4 \) (mod 7). Hence there remain the cases

\[ T_0 = 210m + \mu, \mu = 137, 167, 227, 101, 221, 251. \]

The least \( P \) is now 403, whence \( d \equiv 5 \). Now \( T_0 \equiv m + \mu \) (mod 11). Thus \( T_0 \) is divisible by 11 if \( m \equiv 6, 9, 4, 9, 10, 2 \) (mod 11), respectively. But \( T_k = T_{k-1} + 2k \). Hence if we subtract \( 2k \) from the \( m \) for which \( T_k \equiv 0 \) (mod 11), we obtain the \( m \) for which \( T_k \equiv 0 \) (mod 11). This may be done by counting spaces on square ruled paper. At each point so obtained a hole is punched, thus giving a \( 6 \times 11 \) stencil for \( p = 11 \). The least \( T_0 \) is now 221, whence \( P \equiv 883, g \equiv 8 \).

Similarly, stencils were constructed for \( p = 13, 17, 19, 23, 29 \). After using the first three stencils, it was noted that \( m \equiv 4 \) for each \( \mu \), whence \( T_0 \equiv 941, P \equiv 3763, g \equiv 17 \).

The first 10710 values of \( T_0 \) were examined; to this end \( m \) was given the values \( \leq 1785 \). The use of each stencil excluded more than half of the values left at the earlier stage. After using the stencils for \( p \equiv 29 \), we had left 110 numbers, for each of which \( T_0, \ldots, T_6 \) was verified to be composite. In just four cases were \( T_0, \ldots, T_5 \) all prime. The work, including the making of the stencils, was done in two days.

**Theorem.** For \( 163 < P < 1,500,000 \) there is more than one class of positive primitive quadratic forms of discriminant \(-P\).

For a greater \( P, g \equiv 353 \) and there is more than one class unless \( T_0, T_1, \ldots, T_{353} \) are all primes. The chance that such a case will arise is extremely small. Note that, for \( P \) not exceeding \( 1 \frac{1}{2} \) millions, \( T_0, \ldots, T_{14} \) were shown to be not all prime.