THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-ninth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 29–30, 1911, extending through three half-day sessions. The total attendance was fifty-seven, including the following forty-three members of the Society:

Professor F. Anderegg, Professor W. H. Bates, Professor G. A. Bliss, Dr. H. T. Burgess, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor Arnold Emch, Professor Peter Field, Dr. T. H. Hildebrandt, Dr. Louis Ingold, Professor Kurt Laves, Professor A. C. Lunn, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor J. L. Markley, Professor William Marshall, Mr. R. M. Mathews, Professor Malcolm McNeil, Professor J. C. Morehead, Professor F. M. Morrison, Mr. E. J. Moulton, Professor F. R. Moulton, Professor Alexander Pell, Dr. Anna J. Pell, Professor W. S. Pemberton, Dr. R. E. Root, Mr. J. M. Rysgaard, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor H. E. Slaught, Professor A. W. Smith, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Marion B. White, Professor E. J. Wilczynski, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor L. E. Dickson, Chairman of the Section, presided at the opening session on Friday morning. Professor E. B. Van Vleck presided at the session on Friday afternoon while Professor Dickson was delivering his address on the "History of the representations of numbers as the sum of squares," and Professor E. J. Townsend presided at the session on Saturday morning.

At the business meeting on Saturday morning the following officers of the Section for the year 1912 were elected: Professors D. R. Curtiss, chairman, H. E. Slaught, secretary, and A. L. Underhill, third member of the programme committee.

On Friday noon the members lunched together at the Quadrangle Club, and in the evening they dined together at the same place and spent one of the most enjoyable social occasions in the history of the Section.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "Involutoric circular transformations as a particular case of the Steinerian transformation and their invariant net of cubics."

(2) Dr. R. E. ROOT: "Iterated limits in general analysis."

(3) Dr. ARNOLD DRESDEN: "Reduction of systems of linear differential equations of any order."

(4) Dr. LOUIS INGOLD: "Displacements in a function space."

(5) Professor L. E. DICKSON: "History of the representation of numbers as the sum of squares."

(6) Professor F. R. MOULTON: "Relations of families of periodic orbits in the restricted problem of three bodies."

(7) Professor L. E. DICKSON: "Note on Waring's theorem."

(8) Professor L. E. DICKSON: "Uniqueness of division in Cayley's algebras with eight units."

(9) Professor J. B. SHAW: "On differential invariants."

(10) Professor E. J. WILCZYNSKI: "On some geometric questions connected with the problem of three bodies."

(11) Professor PETER FIELD: "On Coulomb's laws of friction."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space."

(13) Mr. H. F. VANDIVER: "Theory of finite algebras."

(14) Dr. ARNOLD DRESDEN: "Note on the second variation; Jacobi's equation and Jacobi's theorem in the calculus of variations."

(15) Professor G. A. MILLER: "Gauss's lemma and some related group theory."

(16) Professor R. D. CARMICHAEL: "On a class of linear functional equations."

(17) Professor R. D. CARMICHAEL: "On the theory of the gamma function."

Mr. Vandiver's paper was communicated to the Society through Professor Dickson. In the absence of the authors, the papers of Mr. Vandiver, Dr. Bill, Professor Miller, and Professor Carmichael were read by title. Abstracts of the papers, except the historical paper by Professor Dickson, follow below in the order of the list above.

1. It is well known that with every Steinerian, or involutoric quadratic transformation in a plane is associated an invariant net of cubics passing through the fundamental quadruple of

the transformation and its diagonal points. Conversely, every plane cubic admits of an infinite number of birational transformations into itself, which form a continuous group. The involutoric circular transformation in a complex plane

$$z' = \frac{az + b}{cz - a}$$

is a particular case of a Steinerian transformation, and it is the purpose of Professor Emch's paper to establish this equivalence and the principal properties of the two nets of cubics which remain invariant in the transformations of the first and second kind, respectively.

2. The paper by Dr. Root is essentially an amplification, together with some applications, of a method for the investigation of iterated limits outlined in his paper at the April meeting of the Society. The method involves a generalization of the idea of neighborhood of an element, in the sense that a neighborhood of an element is a subclass of the range class specially related to the element. This relation of subclass to element is taken as undefined, and as the basis of the system of postulates. The character and form of the postulates are determined largely by two fundamental requirements: first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of systems. The theorems on functions are, for the most part, direct analogues or generalizations of well-known theorems on multiple sequences and multiple and iterated limits of functions of real variables. Certain special features pertain, however, to the presence of ideal elements. The general theory is found to be available for functions on any range, subject to the Frechet notion of voisinage, or to properly conditioned K_1 and K_2 relations of the type used by E. H. Moore.

3. The purpose of Dr. Dresden's paper is to derive conditions under which a system of n ordinary linear differential equations in $n + 2$ variables and of order p may be reduced to a system of the same type in $n + 1$ variables.

If the given equations are of the form

$$(1) \qquad \sum_{j=1}^{n+1} \sum_{l=0}^{p} a_{ilj} x_j^{(l)} = 0 \qquad (i = 1, \ldots, n),$$

where a_{ilj} are functions of t of class $C^{(l)}$ on a range (t_1t_2), we seek to determine conditions on the coefficients a_{ilj}, which will make it possible to write equations (1) in the form

$$\sum_{k=1}^{n}\sum_{l=0}^{p} A_{ilk}w_k^{(l)} = 0 \qquad (i = 1, \ldots, n),$$

where

$$w_k = \sum_{j=1}^{n+1} \alpha_{kj}x_j.$$

The conditions arrived at are expressible as the vanishing of determinants of order $n + 1$ in which the matrix a_{ipj} ($i = 1, \ldots, n; j = 1, \ldots, n + 1$) forms the first n columns, the last column being formed by sums of derivatives of various orders of the coefficients a_{ilj}.

4. In the first part of Dr. Ingold's paper a set (φ) of three mutually orthogonal unit vectors φ_1, φ_2, φ_3, is considered in relation to any set (ψ) of unit vectors ψ_1, ψ_2, ψ_3 (not necessarily orthogonal) into which the first can be transformed linearly. The vectors of both sets are regarded as functions of any number of parameters $u_1, u_2, \ldots$.

The projections of the derivatives $\partial\phi_i/\partial u_l$ upon the vectors φ_j are denoted by $P_{ij}^{(l)}$; similarly the projections of the derivatives $\partial\psi_i/\partial u_l$ upon the vectors ψ_j are denoted by $Q_{ij}^{(l)}$. A set of fundamental relations (R) is obtained connecting the quantities $P_{ij}^{(l)}$ with the $Q_{ij}^{(l)}$. A special case is the set of equations of Darboux* satisfied by the two-parameter rotations of a rigid system.

If the sets of vectors (φ), (ψ) contain only two vectors each, the corresponding relations (R) still hold and in a special case reduce to a single equation for the two-parameter rotations of a rigid plane tangent to a given surface.

In the second part of the paper the corresponding developments are obtained for a set of n mutually orthogonal normed functions of x ($\alpha \leqq x \leqq \beta$) and any number of parameters, $u_1, u_2, \ldots$, say $\varphi_1(x; u_1, u_2, \ldots)$, $\varphi_2(x; u_1, u_2, \ldots), \ldots$, $\varphi_n(x; u_1, u_2, \ldots)$. Quantities analogous to the $P_{ij}^{(l)}$, $Q_{ij}^{(l)}$ are obtained and a set of relations analogous to the relations (R) is found connecting them. Special cases of this set are generalizations of

* Darboux, Théorie générale des Surfaces, vol. 1, p. 49.

(1) The extensions to n dimensions and any number of parameters of Darboux's equations for the rotations of a rigid system.*
(2) Equations for the rotations of any rigid system moving tangent to any space.
(3) The Darboux extensions of the Lamé conditions for a triply orthogonal system of curves and surfaces in ordinary space.†

6. The theories of a number of classes of periodic solutions of the restricted problem of three bodies have been developed in papers presented to the Society by Professor Moulton in the last ten years. The present paper defines some additional periodic solutions consisting of orbits having multiple loops, examines the character of the changes through certain critical forms, and then takes up the problem of following out the analytic continuity, in the sense of analytic functions, of all the orbits with respect to the various parameters upon which the solutions depend, and of tracing the connections of the several families. Omitting reference to the nature of the branching and the definitions and qualifications necessary in some statements for complete precision, a few of the theorems on which the discussion depends are the following:
(1) Six orbits having real periods and single loops branch from one of zero dimensions at each of the finite masses, of which three are direct and three retrograde. In the vicinity of the origin only one each of the direct and of the retrograde orbits is real. They are symmetrical with respect to the x-axis.
(2) Six orbits having real periods and single loops branch from infinity, three of which are direct and three of which are retrograde with respect to fixed axes, while all are retrograde with respect to rotating axes. When the orbits are sufficiently large only one each of those which are direct and retrograde is real. They are symmetrical with respect to the x-axis.
(3) There is a geometrically single family of three-dimensional orbits branching from each of the five points of libration.

* These extensions are given by Hatzidakis, "Displacements depending on one, two, . . ., k parameters in a space of n dimensions." *American Journal*, vol. 22 (1900), pp. 154–184.

† Darboux, Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, Book II, Chapter I.

Those branching from the collinear libration points are symmetrical with respect to the xy and xz-planes; those branching from the equilateral triangular points are symmetrical only with respect to the xy-plane unless the finite bodies are equal, when they are symmetrical also with respect to the yz-plane.

(4) There is a geometrically single two-dimensional family branching from each of the collinear libration points in which the motion is retrograde with respect to the points.

(5) There are two-dimensional orbits associated with the equilateral triangular libration points falling under a number of cases depending upon the ratio of the finite masses and the parameter employed.

(6) No two finite loops can unite at only a single point.

(7) Single loops develop when, and only when, an orbit passes through a form of ejection or acquires a cusp on a surface of zero relative velocity.

(8) Multiple loops develop when, and only when, an orbit has a corner at a collinear libration point.

(9) If a direct orbit passes through a form of ejection or a cusp form it acquires a loop; a retrograde orbit under the same circumstances loses a loop.

(10) Ejection forms and cusp forms of periodic orbits can disappear only by uniting with other similar forms.

(11) Isolated periodic orbits do not exist.

(12) Three direct orbits have infinite periods; no retrograde orbits have infinite periods.

7. This paper by Professor Dickson gives an algebraic proof of the existence of an identity expressing any elementary symmetric function of $x_1^2, \ldots, x_e^2$ of degree $2m$ in the x's as a linear function with rational coefficients of M powers, with exponent $2m$, of linear functions of $x_1, \ldots, x_e$ with integral coefficients, where M and all the coefficients depend upon m and e alone.

8. In the second paper by Professor Dickson, it is pointed out that the algebras given by Cayley (*Philosophical Magazine*, volume 26 (1845), page 210; *American Journal of Mathematics*, volume 4 (1881), page 293) are equivalent to that with the units $1, e_1, \ldots, e_7$, where

$$e_i^2 = -1, \quad e_ie_j = -e_je_i. \qquad (i,\ j = 1, \ldots, 7;\ i \neq j),$$

$$e_1e_2 = e_3, \quad e_1e_4 = e_5, \quad e_1e_6 = e_7, \quad e_2e_5 = e_7,$$

$$e_3e_4 = e_7, \quad e_3e_5 = e_6, \quad e_4e_2 = e_6,$$

with 14 equations obtained by permuting the subscripts cyclically. It is now shown that right and left hand division, except by zero, is always possible and unique in this algebra. This is done by expressing the general element as a linear function of e_2, e_4, e_6, with coefficients B, C, $\ldots$, linear in e_1, and applying

$$e_jB = \bar{B}e_j, \quad (Be_j)(Ce_j) = (B\bar{C})e_j^2, \quad (Be_j)(Ce_k) = (\bar{B}\bar{C})(e_je_k),$$

for $j, k = 2, 4, 6;\ j \neq k$. Here $\bar{B} = r - se_1$ if $B = r + se_1$.

9. This is the third paper by Professor Shaw along this line, the other two having been presented before the Chicago Section at the meetings in December, 1910, and April, 1911. The method previously presented is now generalized to the case of space of n dimensions. It is shown in a suitable notation that the curvatures of a region of $n - m$ dimensions are expressible as the scalar invariants of the operator which is the generalization of the quaternion $S\,()\,\nabla, \sigma$. The various differential parameters are treated in the same manner. Finally the relation to the Maschke symbols is pointed out.

10. Two years ago Professor Wilczynski showed in a paper read before the Chicago Section how certain geometric problems connected with the problem of three bodies might be formulated, and how a number of these led to the question whether isosceles solutions exist. In the present paper this question finds a complete answer. If no two of the three masses are equal, the only isosceles solutions are the well-known equilateral solutions of Lagrange and its limiting cases. If two of the masses are equal, there exist two essentially distinct types of isosceles solutions.

11. Coulomb's laws of friction state that when one body slides over another the frictional force is (1) proportional to the normal pressure, (2) independent of the relative velocity, (3) independent of the area of the surfaces which are in contact. Painlevé in his Leçons sur le Frottement has shown that if

we assume these laws, peculiar situations may arise in case the problem is of such a nature that the normal pressure is a function of the coefficient of friction. Professor Field in this paper treats by graphical methods a problem of this kind which has been treated analytically by Painlevé and others. The paper will be published in the *Zeitschrift für Mathematik und Physik*.

12. This is the first of a series of papers, which Dr. Bill purposes to present to the Society giving the results obtained by applying to non-euclidean geometry the methods of analysis used by Professor Eduard Study for euclidean space. Starting from certain fundamental identities existing between the absolute invariants of the group of motions, the author arrives, in the first part of his paper, at a complete classification of the analytic curves of non-euclidean space. The second part of the paper is taken up with the derivation of the Frenet-Serret formulas for regular curves.

13. The theory of finite fields, due to Moore (Mathematical Papers, Chicago Congress, 1893, pages 208–242), may be regarded as a generalization of the arithmetical theory of congruences with prime moduli. In the present paper Mr. Vandiver examines an algebra which is an extension of the finite field, and which is defined as follows:

A set of s distinct elements forms a finite algebra of order S if the elements can be combined by addition, subtraction and multiplication, these operations being subject to the commutative and associative laws of elementary algebra, and if the resulting sum, difference, and product be each uniquely determined as an element of the set. Also there must exist in the set at least one element u_k such that the equation $u_k x = u_0$ (u_0 being the zero element) has the unique solution $x = u_0$.

In this algebra division is not always possible, and if possible, may give more than one quotient. Some of the elementary properties of a general finite algebra are given. Elements are classified as units and non-units. Prime and composite non-units are defined. It would appear that the theory includes as particular phases all arithmetical congruence theories, in particular, when the congruences considered have composite moduli.

14. In Dr. Dresden's note, a method is given for the discussion

of Weierstrass's treatment of the second variation, Jacobi's equation, and Jacobi's theorem in the calculus of variations for the case of an integral of the form $\int F(x, y, x', y')dt$. The method is based on the reduction of differential forms and unifies these three subjects; it is, moreover, immediately extensible to cases where the integrand contains more unknown functions, as an illustration of which the space problem is taken up.

15. According to Gauss's lemma any number m which is not divisible by the prime number p is a quadratic residue or a quadratic non-residue of p according as the series m, $2m$, $\cdots$, $\frac{1}{2}(p - 1)m$ includes an even or an odd number of numbers whose least absolute residues (mod p) are negative. The main objects of Professor Miller's paper are to exhibit the setting of this lemma in the theory of abelian groups and to show how readily it can be deduced from this theory.

A set of operators of a group G is called a complete set for the nth powers if the nth powers of these operators give all the different nth powers of the operators of G and if no two operators of the set have the same nth power. The products obtained by multiplying all the operators of a complete set for the nth powers of an abelian group by any operator of the group constitute a complete set for the nth powers. The h complete sets obtained by multiplying any one such complete set by the different operators of the abelian group whose orders divide n are called complementary sets. These complementary sets involve every operator of the abelian group once and only once. If all the numbers in any complete set for squares (mod p) of the numbers $1, 2, \cdots, p - 1$ are multiplied by any one α of these numbers, then α is a quadratic residue or a quadratic non-residue of p according as an even or an odd number of these products occur in the set which is complementary to the given set for squares. This theorem clearly includes Gauss's lemma.

16. It is a problem of great difficulty (and one which is not completely solved in any extensive class of cases) to determine the general solution of a functional equation which involves two operations, as in the case of partial differential equations or mixed equations. The object of the present paper is to investigate the properties of the solutions of an equation belonging

to this general class; namely, an equation of the form

$$f(x+\omega_1+\omega_2) + \varphi(x)f(x+\omega_1) + \psi(x)f(x+\omega_2) + \chi(x)f(x) = 0,$$

certain restrictions being placed on the coefficients $\varphi(x)$, $\psi(x)$, $\chi(x)$. Such an equation involves the two operations of replacing x by $x + \omega_1$ and by $x + \omega_2$. The general theory of these equations throws light on the nature of the solutions of various functional equations involving two operations.

Among the results which Professor Carmichael obtains are the following: The arbitrary elements in the solution of the equation are doubly periodic functions of periods ω_1 and ω_2. Regions in which the solutions may be chosen arbitrarily are found; and by means of such regions it is shown that an infinite number of arbitrary elements must enter into a general solution of the equation. Analytic solutions involving simply periodic functions (some of period ω_1 and others of period ω_2) are obtained. A means is also given for expressing linearly in terms of particular solutions of this type (the multipliers being doubly periodic functions) any one of a large class of quasi-general solutions.

17. From the general theory of difference equations (see papers by Carmichael and by Birkhoff in the *Transactions*, volume 12) it follows that the equation $g(x + 1) = xg(x)$ has two solutions $g_1(x)$ and $\bar{g}_1(x)$ having the property that each of the limits

$$\lim_{x=+\infty} g_1(x)x^{-x+\frac{1}{2}}e^x \quad \text{and} \quad \lim_{x=-\infty} x^{-x+\frac{1}{2}}e^x$$

exists and is 1, where x goes to infinity along the axis of reals in the positive sense in the first case and in the negative sense in the second case. Moreover,

$$\bar{g}_1(x) = (1 - e^{2\pi x\sqrt{-1}})g_1(x).$$

Also, any solution $g(x)$ is uniquely determined if $g(1)$ is assigned (different from zero) and

$$\lim_{x=+\infty} g(x)x^{-x+\frac{1}{2}}e^x$$

exists. It is natural to choose for the definition of $\Gamma(x)$ one that associates it most intimately with this general theory. Accordingly, Professor Carmichael defines a first and a second gamma function, $\Gamma(x)$ and $\bar{\Gamma}(x)$, as those solutions of the equation $g(x+1) = xg(x)$ which have the properties that

$$\Gamma(1) = 1, \qquad \bar{\Gamma}(x) = (1 - e^{2\pi x\sqrt{-1}})\Gamma(x),$$

and

$$\lim_{x=+\infty} \Gamma(x)x^{-x+\frac{1}{2}}e^{x} \quad \text{exists.}$$

If one starts from these definitions and makes use of the general theory of linear homogeneous difference equations of the first order, the fundamental properties of $\Gamma(x)$ and $\bar{\Gamma}(x)$ are readily obtained. The theory, as worked out recently by Professor Carmichael in his academic lectures, is decidedly simpler and more elegant than the usual theory of the gamma function, as developed, for instance, in Nielsen's Handbuch.

H. E. SLAUGHT,
Secretary of the Section.

AN IDENTICAL TRANSFORMATION OF THE ELLIPTIC ELEMENT IN THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 29, 1911.)

THIS paper is based upon a formula published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures. The formula is, after correcting slight misprints in Biermann's pamphlet,

$$(1) \quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A \cdot R(x_0)}.$$

F is the solution of

$$(2) \quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant