following remark, which is stated without proof. The rôle of the adjoint homogeneous equation in the ordinary theories* is here taken by a pure homogeneous linear integral equation together with linear integral auxiliary conditions

$$(1') \qquad v(x) = \int_a^b v(s)K(s, x)ds; \quad \int_a^b v(x)k_i(x)dx = 0$$

$$(i = 1, 2, \cdots, m);$$

but the numbers of independent solutions of (1) and (1') need not, as in the usual cases, be the same. For example, there can never be more (and will in general be fewer) than n linearly independent continuous solutions of (1'), while there may under certain conditions be as many as $m + n$ linearly independent continuous solutions of (1).

CORNELL UNIVERSITY,
December 22, 1911.

NOTE ON THE GRAPHICAL SOLUTIONS OF THE FUNDAMENTAL EQUATIONS IN THE SHORT METHODS OF DETERMINING ORBITS.

BY PROFESSOR A. O. LEUSCHNER AND MR. B. A. BERNSTEIN.

(Read before the San Francisco Section of the American Mathematical Society, October 28, 1911.)

THE chief difficulty in the determination of preliminary orbits of new comets or asteroids arises from the fact that the geocentric or heliocentric distance of the new body enters into the problem as an unknown quantity. In the older methods the heliocentric distance often is assumed to be unity as a first approximation in the determination of comet orbits. The average heliocentric distance of the known asteroids is generally chosen as a first approximation for new asteroids. The true distances are then evaluated by laborious approximations.

In the short methods proposed by Professor Leuschner it is possible to make a direct determination of the geocentric dis-

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* E. g., for the pure linear integral equation, the equation with transposed kernel; for the system of linear algebraic equations, the system with transposed (or conjugate) matrix.

tance at the middle date of the three underlying observations to as many decimals as may seem desirable. This is accomplished by a graphical solution in the determination of parabolic or circular orbits and by means of a table of geocentric distances for general orbits.

For the parabola or circle the fundamental equation is of the form

$$f(z) = (z - p')^2 + q'^2 - \frac{h}{[(z - c)^2 + s^2]^{\frac{3}{2}}} = 0,$$

where the unknown z represents the ratio of the geocentric distance of the body to the geocentric distance of the sun and where all other quantities are known constants. The roots of the equation are given by the intersections of the parabola

$$y = (z - p')^2$$

with the asymptotic curve

$$y = \frac{h}{[(z - c)^2 + s^2]^{\frac{3}{2}}} - q'^2.$$

The distance between the axes of symmetry of the two curves is $c - p'$, reckoned from the axis of the parabola. The equations may be written

$$y = z'^2, \quad \eta = \frac{h}{(\zeta^2 + s^2)^{\frac{3}{2}}},$$

where $z' = z - p'$, $\zeta = z - c$, $\eta = y + q'^2$. The equation of the asymptotic curve may also be written

$$\zeta = s \tan \theta, \quad \eta = \frac{h}{s} \cos \theta.$$

In the astronomical problem only the positive real roots of $f(z) = 0$ need to be considered. There are either one or three positive real roots. The portions of the asymptotic curve near its intersections with the parabola may be conveniently plotted (Fig. 1) without the aid of a trigonometrical table, as follows: Draw a circle with radius h/s about the origin $\eta = 0$, $\zeta = 0$ as center and draw the line $\eta = s$. Draw any vector from the origin making an angle θ with $\zeta = 0$. Through the intersection M of vector and circle draw MM'

parallel to $\eta = 0$ and through the intersection N of vector and $\eta = s$ draw NN' parallel to $\zeta = 0$. Then for the intersection P of MM' and NN' we have

$$\zeta = s \tan \theta \quad \text{and} \quad \eta = \frac{h}{s} \cos \theta$$

and, therefore, P is a point on the asymptotic curve.

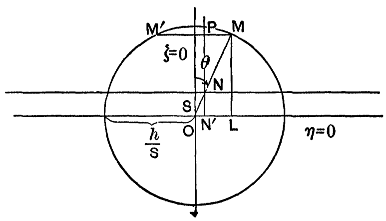

FIG. 1.

This construction was independently suggested by Mr. B. A. Bernstein. In practice the parabola $y = z'^2$ is first placed in proper position (with its vertex at $\zeta = p' - c$, $\eta = q'^2$ and its axis parallel to $\zeta = 0$) and a point P is located on either side of an intersection. The values of ζ corresponding to the intersections may then be read off and finally the corresponding values of z are given by $z = \zeta + c$. In locating a point P it is not necessary actually to draw either the vector or MM' or NN', but P may be located by means of rulers held in the proper position. In case of more than one solution the corresponding roots z may thus be obtained at one and the same time.

In the solution of an orbit without hypothesis regarding the eccentricity the equation is of the form

$$(z^2 - 2c + 1)^3(z - m)^2 - m^2 = 0.$$

There may be either one or two positive real roots which may be taken from a table with ψ and $1/m$ as arguments where $\psi = \cos^{-1} c$ and m are known constants. The table is an extension of Table XIIIa of von Oppolzer, Lehrbuch der Bahnbestimmung, volume 1.

In case of three parabolic solutions, the physical solution is determined by identification with a general solution. The mathematical parabolic solutions may thus be rejected in the course of computation. The only known cases with three parabolic solutions are comet Cruls of 1882 and comet 1910*a*. Graphical solutions for these comets have been made by Mr. E. S. Haynes, John H. Pitman, and Miss S. Levy.

An application of the foregoing methods of solution for comet 1910*a* has been published by Professor Leuschner in the *Lick Observatory Bulletin,* No. 197. A full account with the tables is being printed in the *Publications of the Lick Observatory,* volume 7.

Mr. Bernstein gives a purely geometric solution of the fundamental equation that enters into the *general* solution of an orbit. He proceeds as follows:

From the equation

$$(z^2 - 2c + 1)^3(z - m)^2 - m^2 = 0 \tag{1}$$

it follows that

$$z - m = \frac{m}{(z^2 - 2c + 1)^{\frac{3}{2}}},$$

where only the positive sign before the radical need be considered.

Making the substitution $x = z - c$, and writing $1 - c^2 = s^2$ (since $c < 1$), the above becomes

$$x + c - m = \frac{m}{(x^2 + s^2)^{\frac{3}{2}}}. \tag{2}$$

The roots of this equation are got from the intersections of the curve

$$y = \frac{1}{(x^2 + s^2)^{\frac{3}{2}}} \tag{3}$$

with the straight line

$$my = x + c - m. \tag{4}$$

If we make the substitution $x = s \cot \theta$, we get

$$\begin{aligned} x &= s \cot \theta, \\ y &= \frac{1}{s^3} \sin^3 \theta = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta \end{aligned} \tag{5}$$

as the parametric equations of the curve represented by (3). This curve can now be constructed geometrically (Fig. 2) as follows:

Draw a line $\overline{A'A}$ parallel to the x-axis at a distance s above it. With the origin O as center construct two circles having for radii $1/4s^3$ and $3/4s^3$ respectively. Let θ be any angle which a line $\overline{OQ}$ makes with $\overline{OX}$, the positive direction of the x-axis, and let $\overline{OQ}$ meet $\overline{A'A}$ in R and the circle of radius $3/4s^3$ in Q. Draw the angle $XOT = 3\theta$, and let the terminal

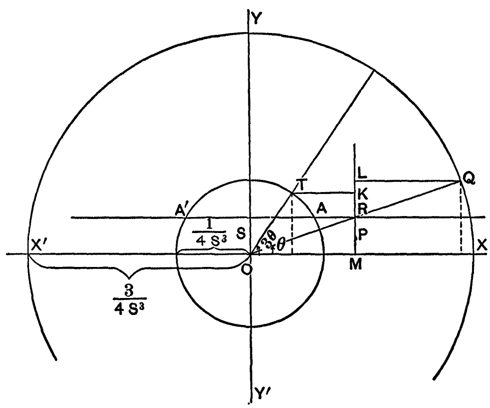

FIG. 2.

side of this angle meet the circle of radius $1/4s^3$ in T. Draw $\overline{RM}$ perpendicular to $\overline{OX}$. From T and Q drop perpendiculars to $\overline{RM}$, meeting the latter in K and L respectively. On $\overline{MR}$ lay off $\overline{MP} = \overline{KL}$. Then P is a point on the required curve.

For

$$x = \overline{OM} = s \cot \theta$$

and

$$y = \overline{MP} = \overline{KL} = \overline{ML} - \overline{MK} = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta$$

We thus obtain the geometric construction of the curve represented by equation (3). Its intersections with the line given by (4) are then easily found, and hence equation (2), and therefore (1), are solved.