THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-seventh regular meeting of the Society was held in New York City on Saturday, February 24, 1912. The attendance at the two sessions included the following forty-one members:

Dr. F. W. Beal, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor E. W. Brown, Professor A. B. Coble, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Dr. H. B. Curtis, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor C. C. Grove, Professor H. E. Hawkes, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Professor J. H. Maclagan-Wedderburn, Dr. E. J. Miles, Dr. H. H. Mitchell, Mr. F. S. Nowlan, Professor W. F. Osgood, Mr. E. S. Palmer, Dr. H. W. Reddick, Dr. J. E. Rowe, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor Henry Taber, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor A. H. Wilson, Professor J. W. Young.

The President of the Society, Professor Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. J. W. Alexander, Princeton University; Mr. A. A. Bennett, Princeton University; Professor J. G. Coffin, College of the City of New York; Professor G. H. Cresse, Middlebury College; Mr. C. R. Dines, Dartmouth College; Professor H. E. Jordan, University of Kansas; Mr. F. S. Nowlan, Columbia University; Professor C. W. Watkeys, University of Rochester. Eight applications for membership in the Society were received.

Announcement was made of the recent gift by Dr. Emory McClintock, second President of the Society, of over four hundred valuable mathematical books to the Society's Library. The gift also includes a large number of pamphlets and reprints of important mathematical papers. The grateful acknowledgments of the Society were tendered to Dr. McClintock by resolution.

The following papers were read at this meeting:

(1) Mr. S. A. JOFFE: "Sums of like powers of natural numbers."

(2) Professor G. A. MILLER: "Second note on the groups generated by operators transforming each other into their inverses."

(3) Dr. S. LEFSCHETZ: "On remarkable points of curves."

(4) Dr. S. E. URNER: "Certain singularities of point transformations in space of three dimensions."

(5) Professor A. B. COBLE: "The characteristic theory of the odd and even theta functions as related to finite geometry."

(6) Dr. H. H. MITCHELL: "Some quaternary groups with particular prime moduli."

(7) Dr. J. E. ROWE: "The undulation and cusp invariants of the R^n."

(8) Professor W. F. OSGOOD: "A necessary and sufficient condition that a single-valued function in a projective space be rational."

(9) Dr. DUNHAM JACKSON: "On the convergence of the development of a continuous function according to Legendre's polynomials."

(10) Mr. K. P. WILLIAMS: "The solutions of non-homogeneous linear difference equations and their asymptotic forms."

(11) Dr. E. J. MILES: "Note on the isoperimetric problem with discontinuous integrand."

(12) Dr. DUNHAM JACKSON: "On approximation by trigonometric sums and polynomials."

(13) Dr. J. E. HODGSON: "Orthocentric properties of the plane directed n-line."

Mr. Williams was introduced by Professor Birkhoff. Dr. Hodgson's paper was communicated to the Society by Professor Morley. In the absence of the authors the papers of Professor Miller, Dr. Lefschetz, Dr. Urner, and Dr. Hodgson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By the use of the identity

$$2x(e^{2x} + e^{4x} + \cdots + e^{2nx}) = e^{(2n+1)x} \cdot \frac{2x}{e^x - e^{-x}} - \frac{2x}{1 - e^{-2x}}$$

Mr. Joffe expresses the sum of the odd, $(2k - 1)$th (even,

$2k$th) powers of the first n positive integers as an integral function of $2n + 1$ (even, of order $2k$; odd, of order $2k + 1$). Noticing then that any even function of $2n + 1$ is an integral function of $n^2 + n$, he establishes the following theorem: The sum of the odd $(2k - 1)$th powers of the natural numbers $1, 2, \cdots, n$ is an integral function (of order k) of $n^2 + n$, with fractional coefficients. For the sum of even powers there is established a similar theorem, but in this case the integral function of $n^2 + n$ is multiplied by $2n + 1$. The general form of the coefficients in both these cases is rather complicated; a few of the initial and terminal coefficients are simplified and are expressed in terms of Bernoullian numbers. Tables are appended giving the numerical values of the coefficients for all the functions corresponding to the first twenty-five powers.

2. Professor Schur recently called attention to several errors in a brief note by Professor Miller entitled "Groups generated by operators which transform each other into their inverses," [cf. *Jahrbuch über die Fortschritte der Mathematik*, volume 40 (1911), page 189]. In the present note Professor Miller develops the theory of these groups much more completely and observes that the true theorems relating to the points in question are even more general than the false ones which they aim to replace. Among these are the following: There is one and only one group of order 2^n, $n > 2$, which can be generated by a set of m operators such that each of them transforms each of the $m - 1$ remaining ones into its inverse and that no two of them are commutative. When $n = 3$ this is the quaternion group, and when $n = 4$ it is the Hamiltonian group of order 16, but it is not Hamiltonian for any larger value of n. It contains two and only two invariant operators when n is odd, and when n is even it involves exactly four such operators. These constitute the cyclic group of order 4 when n is of the form $4k + 2$ and only then.

All the operators of this group of order 2^n have orders which divide 4, and all these operators of order 4 have a common square. This square generates the commutator subgroup. When n is of the form $4k + 2$ exactly half of the operators of the group are of order 4, since the product of an invariant operator of order 4 and an operator of order 2 or 4 is of order 4 or 2 respectively in the group under consideration. The number of the operators of order 2 for the other values of n can be readily stated in terms of general formulas.

3. Dr. Lefschetz's paper is an attempt to precise some notions relative to remarkable points of curves. Given a plane curve the arc of which is everywhere analytic, and a discrete aggregate of points and lines, curves of a given order m through λ of the points and tangent to μ of the lines will in general have a contact of order at most k with the curve at an arbitrary point. For certain points however the contact will be of order $k + 1$, and these are defined as remarkable points. This definition is also extended to points of contact of multiply tangent curves, and to the similar points obtained by consideration of the reciprocal polar of the system with respect to any conic. It is then shown that, when m, λ, μ vary, the points defined belong to a discrete aggregate on the curve, and that the aggregate of lines of the plane that meet the curve in none of the remarkable points has the power of the continuum.

4. Dr. Urner discusses the behavior of a point transformation at a point where its Jacobian vanishes, attention being given chiefly to the transformation of contact of curves and surfaces. The notion of order of singularity is developed, and criteria are given for its determination.

5. A glance at the formulas for the integer transformation of the periods of the odd and even theta functions of p variables shows that the period and theta characteristics are transformed linearly and homogeneously under a collineation group in the finite domain S_{2p-1}, mod 2, which has an invariant null system. The transformations of the period characteristics contain the coefficients of the integer transformation linearly, i. e., they are transformed like the points (or their null S_{2p-2}'s) of the S_{2p-1}. The transformations on the theta characteristics contain the coefficients to the second degree but not homogeneously. Since $c^2 \equiv c$, mod 2, homogeneity can be effected and the theta characteristics are then transformed like a system of 2^{2p} quadrics in S_{2p-1}—precisely those quadrics whose polar systems coincide with the given null system. The even and odd characteristics correspond to quadrics of different type, containing respectively

$2^{p-1}(2^p+1)-1$ or $2^{p-1}(2^p-1)-1$ points. The object of Professor Coble's paper is to show that the entire theory of the dependence of these characteristics can be regarded as an image of the finite geometry and can be established with great facility from this point of view.

6. In a study of the finite quaternary linear groups with ordinary coefficients, some particular quaternary modular groups were noticed by Dr. Mitchell. A group of order 40,320 with the coefficients of the transformations in the $GF(3^2)$ was found to exist. This was shown to have a self-conjugate subgroup of order 20,160, holoedrically isomorphic with the ternary simple group consisting of all transformations with coefficients in the $GF(2^2)$ and determinant unity.

A group holoedrically isomorphic with the symmetric group on seven letters and with the coefficients of the transformations in the $GF(7)$ was also found to exist.

7. It is known that the invariants of the R^n are expressible in terms of the three-rowed determinants of the matrix of coefficients of the three n-ics in the parametric equations of the R^n. Dr. Rowe's paper consists of a formal statement of a method by means of which the undulation condition can be expressed as a determinant of order $4(n-3)$ and the cusp invariant as one of order $2(n-1)$, the constituents of these determinants being the three-rowed determinants mentioned above. An especially interesting feature is the relation which exists between the cusp and undulation conditions of the R^5 in the plane and the analogues of these two singularities of the R^{2k+1} in space of k dimensions.

8. Professor Osgood shows that a single-valued function of n complex variables, homogeneous and of dimension 0, which is analytic at every finite point of the space of these variables distinct from the point $(0, 0, \cdots)$, or has at most an unessential singularity, is a rational function of these variables.

9. Following investigations of Lebesgue on the subject of Fourier's series (*Bulletin de la Société Mathématique de France*, 1910), Dr. Jackson's paper is devoted to the study of the order of the approximation to a continuous function given by the partial sum, to terms of the nth degree of the expansion

of the function in series of Legendre's polynomials. It is found that, if $f(x)$ is a function such that the difference between two of its values does not exceed $\omega(\delta)$ when the difference of the corresponding values of the argument does not exceed δ, where $\omega(\delta)$ is any function which approaches zero with δ and satisfies one or two other restrictions, then the error of the approximation referred to, in the interior of the interval $(-1, 1)$, does not exceed in magnitude a quantity of the order of $\omega(1/n) \log n$. In particular, if $\omega(\delta) = \text{constant} \times \delta$ (Lipschitz condition), the upper limit for the error so obtained is of the order of $\log n/n$; and if $\lim_{\delta=0} \omega(\delta) \log \delta = 0$ (Lipschitz-Dini condition), this upper limit approaches zero when $n = \infty$, that is, the series converges. An analogous result is obtained when it is assumed that $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition, the corresponding upper limit being of the order of $\log n/n^k$.

It is further demonstrated by an example that the hypothesis made in the first theorem is not sufficient in itself to ensure any more rapid rate of convergence than that which was actually shown to be attained, so that the theorem may to this extent be regarded as satisfactory.

10. In this paper Mr. Williams considers the system of non-homogeneous linear difference equations

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) + b_i(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $a_{ij}(x)$ and $b_i(x)$ are such that

$$a_{ij}(x) = x^{\mu}\left(a_{ij} + \frac{a_{ij}^{(1)}}{x} + \frac{a_{ij}^{(2)}}{x^2} + \cdots \right),$$

$$b_i(x) = x^{\nu}\left(b_i + \frac{b_i^{(1)}}{x} + \frac{b_i^{(2)}}{x^2} + \cdots \right)$$

$$(i, j = 1, 2, \cdots, n;\ |x| > R).$$

By a direct use of the well known sum formula, obtained by the method of variation of "constants," which has previously been regarded as purely formal, it is shown that in general there exist two simple solutions $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)$ and $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)$.

The first solution is analytic throughout the finite plane, save for possible singularities at the singularities of the functions $a_{ij}(x)$, $b_i(x)$, and the zeros of the determinant $| a_{ij}(x) |$, and points any number of units to the left of these points; the second solution is analytic except at points any number of units to the right of the singularities of $a_{ij}(x)$, $b_i(x)$. These two solutions are shown to be asymptotic, in the right and left half planes respectively, to the series formally satisfying the given system; and one of them maintains its asymptotic form in the remaining half plane at a sufficient distance from the axis of reals.

The method employed to make the sum formula yield these solutions is to replace the sums in one case by an infinite series, and in the other by an appropriate contour integral.

Like results are derived for a single non-homogeneous linear difference equation of the nth order, reducible to such a system.

11. In dealing with the isoperimetric problem of the calculus of variations, where one is asked to find and discuss the properties of curves

$$D: \qquad x = \varphi(t), \quad y = \psi(t)$$

which minimize the integral

$$J \equiv \int F(x, y, x', y')dt$$

subject to the condition

$$K \equiv \int G(x, y, x', y')dt = L,$$

it is customary to assume that the functions F and G are continuous in their four arguments. In Dr. Miles's note some properties of the minimizing curve D are given when the functions F and G are allowed to have a finite discontinuity as the point (x, y) passes through a given curve M.

12. In his thesis (Göttingen, 1911) Dr. Jackson has proved the following theorem: If $f(x)$ is a function which possesses a derivative of order $k - 1$ satisfying a Lipschitz condition throughout a closed interval, then $f(x)$ may be approximately represented in the interval by a polynomial of degree n or lower, with a maximum error not exceeding K/n^k, where k is independent of n. The most interesting special case is that in which $k = 1$ and $f(x)$ itself satisfies a Lipschitz condition. On the assumption that $f(x)$ has the period 2π and

satisfies the conditions stated above for all values of x, it was shown that a precisely analogous theorem holds for the approximation of $f(x)$ by a trigonometric sum of order n or lower, this result being obtainable as a consequence of the preceding. It is now shown that decided simplification in the proof of both theorems may be effected by proving the second directly (this had been done only for $k = 1$) and deducing the first from it.

This method has the further advantage that the numerical constants involved can be computed more conveniently. For example, if $f(x)$ satisfies the condition

$$|f(x_2) - f(x_1)| \leqq |x_2 - x_1|$$

in the closed interval $(0, 1)$, it can be approximately represented in this interval by a polynomial of degree n or lower, with an error which never exceeds $3/n$, for all positive integral values of n. The same line of investigation leads to results in the theory of Fourier's series.

13. There is a theorem that the perpendiculars let fall from the incenters of three out of four lines of given direction upon the remaining line touch a circle. In Dr. Hodgson's paper a circle is obtained for any even number of lines, beginning with four. If we take this circle for any $2n$ out of $2n + 1$ lines, the $2n + 1$ circles touch a line. The question of the reversal of direction of one or more of $2n$ lines is then taken up, and this is followed by the consideration of the configuration of circles arising from four, five, and six lines.

F. N. Cole,
Secretary.

ON THE FOUNDATIONS OF THE THEORY OF LINEAR INTEGRAL EQUATIONS.*

BY PROFESSOR E. H. MOORE.

1. *The Analogous Systems of Linear Equations.*

The theory of linear integral equations, mathematically considered, has its taproot in the classical analogies between

* Address of the Vice-President and Chairman of Section A of the American Association for the Advancement of Science, Washington, December 29, 1911.