out, were the theory of structure and form, the theory of
invariance, the theory of functions as functions, the theory
of inversions. That these can receive a general treatment
we do not doubt, inasmuch as some of them are receiving
such development. In logistic then we find only a very
definite branch of mathematics, and in this volume we have
the most complete treatment of logistic that exists. The
question that many have asked naturally "How far does it
assist in building up synthetic systems of mathematics" is
easily answered. It reaches arithmetic only after one volume
of 666 pages. We would not expect the complete treatise
then to furnish much that would be of a synthetic nature.
Indeed that would be as unreasonable as to expect to build
Eiffel towers and Eads bridges from a study of postulates
and axioms for the foundation of geometry. While design
rests upon these things in a sense, design antedates them just
as language antedates grammar. It is not fair to the book or
its aim to assert that it does nothing synthetic. Its problem
is philosophical and analytical. It does enough if it shows
us what are the characteristic features of reasoning and gen­
eralizes the types of reasoning. In this respect it is scientific
as well as philosophical. It examines the rules of the great
mathematical game. But it does not play the game nor
undertake to teach its strategy.

JAMES BYRNIE SHAW.

DIFFERENTIAL GEOMETRY.

*Vorlesungen üiber Differentialgeometrie.* Von LUIGI BIANCHI.
Autorisierte Deutsche Übersetzung von MAX LUKAT.
Zweite, vermehrte und verbesserte Auflage. Leipzig und

In 1899 Guichard announced (*Comptes Rendus* 128, page
232) without proof the following theorems:

1. Let $M$ be a point of a quadric of revolution $Q$ whose axis
is of length $2a$, $F_1$ and $F_2$ being the foci of $Q$ and $\varphi_1$, $\varphi_2$
the points symmetric to $F_1$, $F_2$ with respect to the tangent plane
to $Q$ at $M$; let $S$ be a surface applicable to $Q$; as $S$ is applied
to $Q$ the points $F_1$, $F_2$, $\varphi_1$, $\varphi_2$ invariably fixed with respect
to the corresponding tangent plane to $Q$ take positions which
are denoted by $F_1', F_2', \varphi_1', \varphi_2'$; the loci of these respective points describe four surfaces for each of which the mean curvature is equal to $1/a$.

II. When the quadric $Q$ is a paraboloid of revolution the loci of the points $F_1'$ and $\varphi_1'$ are minimal surfaces.

The announcement of these theorems marked the beginning of a new epoch in the theory of the deformation of surfaces. During the following years many eminent geometers attacked problems suggested by them, and in particular the deformation of the general quadric. This problem was solved in the admirable memoirs of Bianchi and Guichard which were jointly crowned by the French Academy in 1908. It is in the exposition of this theory and allied questions that the second edition of Bianchi-Lukat's Vorlesungen differs from the first, and is "vermehrt und verbessert." For the sake of brevity we shall refer to the two editions as (I) and (II).

Chapters 21 and 22 of (I) which deal respectively with $N$-dimensional space of constant curvature and hypersurfaces in spaces of constant curvature are omitted from (II), in order to make room for the theory of deformation of quadrics. It is to be regretted that this abridgment was necessary and that the treatise did not appear in two volumes so that this important theory, as well as topics subsequently referred to, could have been included. In view of this omission practically all of the quadratic differential forms which appear in the book involve only two variables and so there is no necessity of developing the theory for $n$ variables as in (I). One finds accordingly that in (II) Chapter 2 is given over to the theory of quadratic differential forms in two variables. This makes the treatment simpler in places, but we cannot see that there was any gain made by the change. There is introduced an algebraic proof of the theorem: Two binary quadratic differential forms with the same constant curvature are always equivalent, and the transformation equations involve three arbitrary constants.

The other changes in the first part of the book are a few additions to the theory of deformation in general. Those double families of lines upon any surface $S$ which become asymptotic lines on one of the deforms of $S$ are called virtual asymptotic lines. The determination of these families of curves requires the solution of two simultaneous partial differential equations of the second order. Applying the theory of the Picard method of successive approximations,
it is shown that any two intersecting analytic curves $C$ and $C'$ on an analytic surface $S$ determine uniquely a double family of virtual asymptotic lines on $S$ to which these curves belong, one to each family, and the deform of $S$ on which these curves are asymptotic lines is analytic. This theory is applied in more detail to pseudospherical surfaces in Chapter 17, but it is similar to the treatment of this particular case given in Chapter 17 of (I).

At the end of Chapter 8 one finds a theorem due to Chief which is of great service in the subsequent discussion. If $S$ is a non-ruled surface applicable to a ruled surface $R$, to the straight lines on $R$ correspond a family of geodesics on $S$, called $g$. Let $a$ be an asymptotic line of either system on $S$ and draw the tangents to the curves $g$ at points $a$, thus forming a ruled surface $R_i$. The theorem of Chief is that $R_i$ is applicable to $S$ in such a way that $a$ remains rigid during the deformation.

In Chapter 17 we begin to find traces of the results growing out of the theorems of Guichard. After developing the theory of Bäcklund transformations of pseudospherical surfaces as found in (I) (in places the treatment is slightly different), the author considers the real transformations obtained by combining conjugate imaginary ones, and more particularly the result of taking two opposite transformations $B_\sigma$ and $B_{-\sigma}$ in applying the "theorem of permutability" (Vertauschbarkeitssatz). Let two such transformations be applied to a pseudospherical surface $S$, and let $S_1$ and $S_2$ be the resulting surfaces and $S'$ the fourth surface which is the transform of these two; the normals to $S_1$ and $S_2$ meet in a point $M_0$ equidistant from these surfaces and likewise the normals to $S$ and $S'$ in a point $M_0'$. The two surfaces $S_0$ and $S_0'$, the respective loci of $M_0$ and $M_0'$, are applicable to the same surface of revolution and are complementary to one another. However, $S_0$ is not a pseudospherical surface as stated on page 483. According as $\sigma$ is zero, real and different from zero, or pure imaginary, the meridian curve of the surface of revolution is different, when the latter surface is real. But for all three types of $\sigma$ the surface $S_0$ is applicable to the surface of revolution of an imaginary ellipse. Thus one is brought incidentally into contact with the problem of deformation of quadrics.

With the exception of the first five pages the material in
Chapter 18 does not appear in (I). By means of a change of
variables the equations of the preceding chapter are given
a form which define a Bäcklund transformation of spherical
surfaces, that is, surfaces applicable to the sphere, with the
difference that all of the first transforms are imaginary.
However, these may be combined into real transformations
as described in the following theorem: If \( S_1 \) is an imaginary
Bäcklund transform of a spherical surface \( S \) by means of a
transformation \( B_{\tau}, \ (\sigma_1 = a + \i b) \), and \( S_2 \) is the transform
of \( S \) by \( B_{\tau'}, \ (\sigma_1 = a - \i b) \), the fourth surface \( S' \) given by the
"theorem of permutability" is a real spherical surface. When
\( b = 0 \), the normals to \( S \) and \( S' \) meet in a point \( M_0 \), and the
normals to \( S_1 \) and \( S_2 \) in a point \( M_0' \); the surfaces \( S_1 \) and \( S_2' \),
the loci of these points, are complementary surfaces and are
applicable to the same prolate ellipsoid of revolution. When
\( b = \frac{1}{2} \pi \), \( S_0 \) and \( S_0' \) are applicable to a hyperboloid of revolution
of two sheets.

The second part of Chapter 18 is given over to the proof of
the two theorems of Guichard stated at the beginning of this
review, and converses of these theorems. In preparation for
this discussion the author considers congruences of spheres,
whose surface of centers \( S_0 \) undergoes a deformation, and also
congruences of lines invariably fixed to a surface which is
being deformed. If one of the sheets of the envelope of spheres
has constant mean curvature as \( S_0 \) is deformed, the same is
true of the other sheet. If two sheets are minimal surfaces,
\( S_0 \) is applicable to a paraboloid of revolution \( P \), the radius of
a sphere being the distance between the corresponding point
on \( P \) and the focus. If the mean curvature of the two sheets
is different from zero and constant, \( S_0 \) is applicable to a pro­
late ellipsoid of revolution \( E \) or a two sheeted hyperboloid
of revolution \( H \), and the radius of a sphere is the distance
between the corresponding point on \( E \) or \( H \) and a focus.

The Bäcklund transformations of a pseudospherical or a
spherical surface \( S \) may be looked upon as transforming the
surface elements of \( S \), that is the points of \( S \) (called the centers
of the elements) and small portions of the tangent planes
about the points, into \( \infty^3 \) surface elements in space which can
be grouped so as to be the surface elements of \( \infty^1 \) surfaces,
the transforms of \( S \). This point of view is not peculiar to
these transformations. As a matter of fact it is thus that
the author sets up the transformations of surfaces applicable
to any quadric in the following very simple and remarkable manner: Let $Q$ be any quadric and $Q_1$ a confocal quadric; let a surface element $f$ have for center $F$, a point of $Q$, and for its plane $\pi$ the tangent plane to $Q$ at $F$; this plane $\pi$ meets $Q_1$ in a curve $C$; take a point $F_1$ of $C$ as the center of the surface element $f_1$ whose plane $\pi_1$ is determined by the line $FF_1$ and one of the rulings of $Q_1$ through $F_1$, it being understood that rulings of the same family are used in setting up the surface element for each point of $Q$; the plane $\pi_1$ is tangent to $Q_1$ at some point of the ruling. The fundamental theorem is the following:

**Theorem (A):** If $Q$ is deformed in any manner into a surface $S$ and in the deformation to each element $f$ there are invariably coupled $\infty^1$ elements $f_1$, obtained by varying the point $F_1$ of $C$ for each point $F$, these $\infty^3$ surface-elements can be correlated into a unique family of $\infty^1$ surfaces $S_1$, each of which is applicable to $S$ and to $Q$.

This transformation which converts $S$ into a surface $S_1$ is called a transformation $B_k$, where the subscript $k$ is a constant determining $Q_1$ among the quadrics confocal to $Q$. The proof of this theorem for the case when $Q$ and $Q_1$ are hyperbolic paraboloids occupies, with certain allied problems, the whole of Chapter 19. The processes are in keeping with the general treatment of problems of differential geometry in the former chapters of the treatise and as a matter of fact it is essentially that followed by Bianchi in his prize memoir. The coordinates of $Q$ and $Q_1$ are given in explicit form in terms of parameters referring to their generators. By straightforward steps the relations between the surface elements $f$ and $f_1$ are established and it is shown that the correlation of the $\infty^8$ elements $f_i$, arising from the $\infty^2$ elements determined by a surface $S_i$ into $\infty^1$ surfaces $S_1$ requires the integration of an equation of the Riccati type. When the linear element of $S_1$ is calculated, it is found that it is not of the same form as that of $S$; consequently $S$ and $S_1$ are not applicable in such a way that the centers of corresponding surface elements correspond in applicability, if at all. Since $Q$ and $Q_1$ are confocal paraboloids of the same family, the orthogonal trajectories of the family establish a correspondence between $Q$ and $Q_1$, namely corresponding points lie on the same trajectory. This is the transformation of Ivory of the one quadric into the other; it plays a very important role in this theory,
as we shall see. Let $S$ be any deform of $Q$ and $S_1$ one of the surfaces defined by the correlation of the displaced surface elements $f_1$. If $M_1$ on $S_1$ corresponds to $F_1$ on $Q_1$, and $M$ on $S$ to the point $F$ on $Q$ which is the Ivory transform of $F_1$, the surfaces $S$ and $S_1$ are applicable with the points $M$ and $M_1$ in correspondence. The proof of this result covers ten pages most of which involves extensive calculation, so that at times the reader's interest lags, but throughout the whole treatment the geometrical ideas are extremely clear. There is no indication of the reason for hitting upon this correspondence of Ivory as the one leading to applicability. Later in the chapter certain properties of the Ivory transformation are derived as a means to the proof of the theorem: If $S_1$ is the transform of $S$ by a $B_k$, then $S$ is a transform of $S_1$ by the same $B_k$; these properties may have led to the suggestion of the trial of the transformation as a good guess, but the reader is left in the dark. The development of the results throughout the chapter is simplified and made more interesting by associating with the surfaces $S$ the ruled surfaces $R$, applicable to $S$, as derived by means of the Chieffi theorem. Among these results one finds this pretty theorem: If the quadric $Q$ rolls upon an applicable ruled surface $R$, each of the congruences $\Gamma, \Gamma$ generated by the lines of the first and second system on a confocal quadric $Q_1$ may be arranged in a unique manner into a simple infinity of ruled surfaces $R_1$, each of which is applicable to $Q$ and to $R$; moreover, the applicability of $R_1$ on $R$ is accomplished by a continuous deformation. The surfaces $R$ and $R_1$ of the foregoing theorem possess the following property in common with the non-ruled deforms of $Q$:

**Theorem (B):** The lines $MM_1$ joining corresponding points on a surface $S$ and a surface $S_1$, resulting from $S$ by a transformation $B_k$, constitute a $W$-congruence of which $S$ and $S_1$ are the focal surfaces.

The remainder of Chapter 19 is given over to the proof of Theorem (B) for the case where $Q$ is a hyperbolic paraboloid. The method is much simplified by use of the results for the ruled surfaces $R$ and $R_1$.

In Chapter 20 Theorems (A) and (B) are established for the case where $Q$ and $Q_1$ are confocal hyperboloids of one sheet. The method followed is similar to that of the preceding chapter with very little abridgment. In fact the
formulas differ only in form, and if anything they are more involved in the present case. Consequently the geometrical results are practically the same in the two chapters. From a historical point of view this fact is very interesting, since the earlier methods which were fruitful in the study of the deforms of paraboloids could not be applied in the case of central quadrics. It should be mentioned, however, that the exceptional case of hyperboloids of revolution leads to a theory of transformation of Bertrand curves analogous to the transformation of Razzaboni.

From the manner of construction of the surface elements \( f_i \) it follows that if \( Q \) is real these elements are real only when \( Q_1 \) has negative curvature. Hence whatever be the type of \( Q \), if only real, the surfaces \( S \) applicable to it admit real transforms \( S_1 \). However, the Ivory transformation makes real points on \( Q_1 \) correspond to imaginary points on \( Q \), when the latter is not a hyperbolic paraboloid or a hyperboloid of one sheet. In this case the applicability of \( S \) and \( S_1 \) is ideal. These cases are treated in Chapter 21. By a slight change of notation the formulas of the two former chapters are used to establish Theorems (A) and (B) for the elliptic paraboloid, the hyperboloid of two sheets, and the ellipsoid. A novel converse result is that any real surface applicable in the ideal sense to a quadric of any of these three types admits of \( \infty^1 \) transforms \( S_1 \) applicable to the same quadric in the real sense. In a like manner there are real surfaces applicable in the ideal sense to the hyperbolic paraboloid and the hyperboloid of one sheet; such a surface is transformable by a \( B_\xi \) into a real surface applicable to the same quadric in the ideal sense. The discussion of these transformations closes with the following "theorem of permutability:"

If \( S \) is a surface applicable to a quadric \( Q \), and \( S_1 \) and \( S_2 \) are two transforms of \( S \) by means of transformations \( B_{k_1} \) and \( B_{k_2} \), where \( k_1 \) and \( k_2 \) are different constants, there is a fourth surface \( S' \), also applicable to \( Q \), which arises from \( S_1 \) and \( S_2 \) by transformations \( B_{k_1'} \) and \( B_{k_2'} \), respectively.

For the proof of this theorem the reader is referred to the third volume of the second Italian edition. Here and in the memoir presented to the French Academy one will find also a treatment of surfaces applicable to the imaginary quadrics. One is inclined to feel that owing to the similarity of treatment of the several cases and in view of the existence of these
details elsewhere much of Chapter 20 and part of Chapter 21 should have been omitted, and the space thus gained should have been used for other material appearing in the second Italian edition which sets forth new ideas. For example, there might have been included the treatment of surfaces with plane or spherical lines of curvature, or an exposition of the Weingarten method of deformation. If the latter had been developed in the manner of the second Italian edition, the ideas and formulas of moving axes would have been introduced, although rather late. We feel that the omission of this operator from the German editions is a serious one. It is to be greatly regretted that both the German and Italian second editions were not rewritten to such an extent that the moving axes could have been used to advantage.

The last three chapters deal with triply orthogonal systems of surfaces. The contents and treatment in the first two of these chapters are essentially the same as in the first edition. By far the greater part of the second chapter deals with confocal quadrics and geodesics on quadrics. This material could have been introduced into Chapter 6 and it would have served as a fine illustration of the general theory thereof set forth. We have always felt that this should have been done in the first editions. In fact we cannot see the reason for concealing this subject matter in a chapter which a beginner is not likely to read. The last chapter of (I) deals with Lamé families of pseudospherical surfaces. In the last chapter of (II) the treatment of spherical surfaces as well is included, together with that of certain surfaces applicable to quadrics.

We were pleased to find a fairly complete index in addition to a table of contents, which appears in (I) alone. The translation is well done, there are very few typographical errors and the bookmaking is up to the standard of the Teubners.

LUTHER PFAHLER EISENHART.