*T can be replaced by three transformations which are one-to-one and analytic both ways, combined with one transformation of the form*

$$x = u, \quad y = v^2.$$

THEOREM XI. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \cdots,$$
$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \cdots,$$

*where the terms quadratic in u and v are not identically zero for either f or $\varphi$, and where these quadratic terms have no common factor, then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0$, $y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0$, $v = 0$ when $x = 0$, $y = 0$.*

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## DARWIN'S SCIENTIFIC PAPERS.

*Scientific Papers.* By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. III, Figures of Equilibrium of Rotating Liquid and Geophysical Investigations, xv + 527 pp.; Vol. IV, Periodic Orbits and Miscellaneous Papers, xviii + 592 pp. Cambridge University Press, 1910, 1911. Royal 8vo.

THE first two volumes of Sir George Darwin's researches have already been reviewed in the columns of the BULLETIN.* They contained papers on the practical and theoretical tidal problems which the oceans present and his earlier attacks on the past history of the earth-moon system. The third and fourth volumes contain his investigations on the relations of fluid masses in rotation about an axis under gravitational forces, on the periodic orbits which a particle can describe when attracted by two bodies of finite masses moving in circular orbits about one another, and a number of papers on other matters.

Of the forms which a single mass of fluid can take when revolving without relative motion about an axis under its own gravitational attraction only, two have long been known. Maclaurin's ellipsoid is one of revolution about the axis of rotation and its eccentricity will have a value which depends

* Vol. 16, pp. 73–78.

on the rate of angular rotation. Jacobi's ellipsoids have three unequal axes. These forms are of little interest unless the motion is stable. As the eccentricity of the Maclaurin ellipsoid increases, the series reaches a point where stability changes to instability. At this place the Jacobian series changes from having three unequal axes to an ellipsoid having the two axes in the equatorial plane equal, that is, to the Maclaurin ellipsoid and, what is more important, from stability to instability.

If the stable portion of the Jacobian series be now followed from this point, there comes a place where the motion changes from stability to instability and at this place it crosses a third series, known as the pear-shaped figures of equilibrium which there degrade into a Jacobian ellipsoid. These pear-shaped figures were first known through a memoir by M. Poincaré which appeared in 1885.

Is it possible to carry the investigation further along the same lines so as to discover figures of relative equilibrium in which, as a certain constant alters, the fluid mass shall separate into two parts, the motion remaining stable? From the point of view of the development of the earth-moon and similar systems, this is the principal question.

While Poincaré was carrying the development to the stage mentioned above, Darwin had already attacked the problem from the other end. In the paper "On figures of equilibrium of rotating masses of fluid" he considers two equal masses revolving near to one another in relative equilibrium and examines the forms they will assume as they approach towards contact. By the use of spherical harmonic analysis he carries the solution to the stage where they are not only in contact but actually overlap. The last is of course a physical impossibility but it leads directly to the possibility of the existence of a figure formed of two masses joined by a narrow neck. The stability question is not, however, settled at this time. Several cases where the two masses are unequal are also considered.

In the attempt to use spherical harmonic analysis, the numerical applications become doubtful when the deviation from a sphere is too great. Darwin therefore undertakes a laborious and detailed investigation into the properties of ellipsoidal harmonics, so that a start may be made with an ellipsoid instead of with a sphere, and that the deviations from the ellipsoidal form may be computed. More than a quarter of Volume III is devoted to this subject, but the space is not extravagant in view of the fact that the papers which follow mainly owe their success to the completeness with which he has developed the theory of these complicated functions and to the methods he has adopted in order that numerical results might be deduced from them.

The first step is a computation to find the critical Jacobian ellipsoid where this series crosses the pear-shaped series. It is found that the axes are nearly in the ratio 8 : 10 : 23, the actual values being given to five significant figures. The ellipsoidal harmonic analysis is then applied in order to find out in what way the pear-shaped figure develops from this critical case. It is shown that the principal changes take place at first in the longest axis. One end of this is lengthened and narrowed, the other shortened and blunted. In the following paper the approximation is carried to a higher order with the the result that the deviation from the ellipsoid is small everywhere except at the narrowed end and it there takes more nearly the form of a protuberance. Sir George Darwin conjectures that the further stage in the development would be the appearance of a furrow at this end, suggesting an ultimate detachment of a small portion of the mass. The general resemblance to a pear is much less striking than in M. Poincaré's conjectural figure drawn without a reduction to numbers.

The chief object of this second paper is, however, the determination of the stability of the series. Sir George Darwin comes to the conclusion that they form a stable series up to some point not determined. He is nevertheless careful to state M. Liapounoff's dissent from this conclusion in spite of the fact that a reexamination of his own work and a repetition of his calculations has not altered the coefficient by which the stability is determined.

In the last paper of volume III "On the figure and stability of a liquid satellite," the problems with which the series of memoirs originally started, as well as other kindred problems, are finally solved. The author opens with a tribute to Edouard Roche whose paper on fluid masses was published in 1847–50 at Montpellier but was neglected for many years. Roche's problem was the determination of the figure and stability of a liquid satellite revolving without relative motion in a circular orbit about a planet which is spherical and rigid.

Sir George Darwin replaces the rigid planet by a mass of fluid which can take the form appropriate for relative equilibrium, but he remarks that the analysis for both is practically the same up to the point where the stability has to be considered. He considers cases of different as well as of equal masses. In both problems the liquid assumes approximately ellipsoidal forms, so that the deformations may be conveniently determined by ellipsoidal harmonic analysis. At the limit of stability of the Roche figure, it is shown that the surfaces of the two bodies are separated by an interval which is not zero. When both the masses are liquid, the series of figures can be continued until they overlap, so that the dumbbell-shaped figure again appears as a possibility. Unfortunately, these figures before junction become very unstable and the author considers that it is inconceivable that a junction by a narrow neck of fluid should render them stable.

Thus the gap between two detached liquid masses and a single mass is not yet filled by stable figures of equilibrium. That one may not draw final conclusions concerning the genesis of a two-body system from this result is sufficiently certain. There are possibilities pointed out by the author—other figures of equilibrium, changes produced by heterogeneity of density, turbulence of motion during the fission,—not yet investigated, any one of which may furnish the desired solution of the problem.

The short paper entitled "The approximate determination of the form of Maclaurin's ellipsoid," published in the *Transactions* of the Society in 1903, is to be recommended as a beginning in a study of the methods employed to reach these conclusions. This problem, which of course admits of an exact solution, is treated by the methods employed in the more complicated cases. As the analysis is comparatively simple and is only carried to the degree of approximation necessary for clearness of exposition, it is easy to follow the several steps and to obtain a comprehensive view of the method.

One further paper on cosmogony is placed in Volume IV. Of this, Sir George Darwin says "I find it very difficult to estimate its value. If it is to be judged by the amount of comment to which a paper gives rise, it is a failure; for it has received but little notice." Perhaps an estimate fairer to its author would be that it was published (1889) somewhat ahead of the time when the need for such an investigation became pressing. Briefly, it consists of an attempt to show that under certain conditions, fully defined, the nebular hypothesis of Laplace will lead to developments in the early history of a planetary system somewhat resembling those produced by the accretion of meteoric matter. While the former of these hypotheses has received a degree of attention commensurate with the time it has been before the scientific public, the latter is comparatively young and has yet to undergo extensive criticism and development before it may take a place beside its older rival. It is not hard to believe that before long this paper may form a foundation for the comparison of the two hypotheses and that the methods used therein may serve as a basis for a reconciliation or a distinction which may lead to the general acceptance of one or the other.

When Dr. G. W. Hill published in 1877 his now famous papers on the motion of the moon, probably neither he nor his early readers imagined that they would open up a new field in the study of problems concerning the motions of particles and bodies. He showed that it was possible, under certain restrictions, to obtain a closed orbit relative to certain moving axes and he gave a series of figures of such orbits for different values of the lunar month. M. Poincaré, taking up the idea, developed it in his essay of 1891 on the problem of three bodies, under the name of the periodic solution. Sir George Darwin, however, had evidently recognized their value before this time, for he made a recommendation to the reviewer to undertake a study of Hill's papers as a beginning likely to lead to results. He himself, however, published nothing on the subject until 1897 when the long paper entitled "Periodic orbits" appeared in the *Acta Mathematica*. Dr. Hill had taken the disturbing body (the sun) of infinite mass moving uniformly round the earth at an infinite distance, and had considered the periodic motion of a satellite of the earth; he finds a portion of a single family of orbits, the moon belonging to one member of the family. Sir George Darwin took the sun to be moving uniformly in a circular orbit at a finite distance and to be of mass equal to ten times that of the earth or planet, and he studies not only the satellite orbits of the third body but also the planetary orbits which are periodic. The main difficulty arises from the adoption of so small a mass of the sun relative to the planet. The ordinary methods of development in series become inapplicable owing either to lack of or to extreme slowness of convergence. He therefore resorted to mechanical quadratures. Starting at an arbitrary place on the sun's radius vector with a given velocity, the orbit is traced piece by piece until it crosses the axis on which it started. If it proves to be reentrant, the periodic orbit is obtained; if not, the starting point is changed and a new orbit computed. After several attempts, by judging the initial point from the previous failures, an orbit is at length obtained which is exactly periodic. There is practically one unknown constant and this has to be determined in each case by the trial and error method just explained. The devices for performing this work accurately, so that errors shall not accumulate, are ingenious.

The determination of the stability is a more difficult question, since it involves the making of an harmonic analysis of the orbit, and the computation of a determinant of many rows and columns, but it is essential and is carried out in each case with complete success. Many orbits belonging to several different families are found. This is pioneer work into a region practically untouched previously and has opened the way to further investigation. In a later paper improved methods and some additional orbits are given.

The remaining papers in the two volumes cover a variety of subjects and several of them are highly suggestive. They include astronomical and cosmogonic investigations, mechanical devices, thrust of a mass of sand, the formation of ripple marks in sand, the treatment of observations, and two statistical papers (1875) on the effects of marriage between first cousins. A detailed mention of them would take us too far. There are also three addresses which one is glad to have in permanent and convenient form, as well as the English version of "The Tides" written for the Encyclopädie der mathematischen Wissenschaften.

The care which the author has bestowed on the editing of his papers has added greatly to the value of the collection. Besides the correction of small errors, calculations have occasionally been repeated and in the cases where he considers that his work has been in error, such portions are omitted and an extract or even a whole paper by another writer who may have furnished the correction is inserted. One is struck too by the generous and full acknowledgment which is given to those from whose papers, letters, or remarks the author has derived any ideas or assistance. The fourth volume completes the list of papers published up to the end of 1910: we hope that as further material accumulates it may be cast into the same form.

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## MATHEMATICAL ECONOMICS.

*Manuel d'Economie politique.* Par VILFREDO PARETO. Paris, Giard et Brière, 1909. 695 pp.

IN the year 1906 Pareto's Manuale di Economia politica con una Introduzione alla Scienza sociale was published at Milan. Three years later a French translation by Alfred Bonnet, with revisions by the author, appeared as volume 38 in the Bibliothèque internationale d'Economie politique. This work, although written by the economist who has most insistently stood for the scientific mathematical method in economics, is not itself primarily mathematical except in spirit. The text to be sure, uses curves freely, is replete with logical keenness, and draws considerably upon mathematical language and upon mechanical, or even thermodynamical, analogy in discussing and illustrating economic equilibrium; but the strictly mathematical treatment, which might be technical to the point of causing some of the world's best literary economists insuperable difficulties, has in every instance been relegated to the long appendix of 133 pages. This arrangement has also been adopted by Irving Fisher in his recent work on The Purchasing Power of Money. So long as only a few students of political economics acquire the necessary knowledge of calculus such a segregation must remain inevitable.

In the first place, as there seems to be no very widespread notion among mathematicians, perhaps even among economists, as to what mathematical economics is and does, it may be well to define a little.

It is clear that the individual as a social unit and the state as a social aggregate require a certain modicum of mathematics, some arithmetic and algebra, to conduct their affairs. Under this head would fall the theory of interest, simple and compound, matters of discount and amortization, and, if