SECOND NOTE ON FERMAT'S LAST THEOREM.

BY PROFESSOR R. D. CARMICHAEL.

In a note printed on pages 233–236 of the present volume of the BULLETIN I have proved the following theorem:

If $p$ is an odd prime and the equation

$$x^p + y^p + z^p = 0$$

has a solution in integers $x$, $y$, $z$ each of which is prime to $p$, then there exists a positive integer $s$, less than $\frac{1}{2}(p - 1)$, such that

\[(1) \quad (s + 1)^p \equiv sp^2 + 1 \mod p^3.\]

Professor Birkhoff has called my attention to the fact that condition (1) may be replaced by the simpler condition

\[(1') \quad (s + 1)^p \equiv sp^2 + 1 \mod p^3,\]

these two conditions being equivalent. Let us define the integers $\lambda$ and $\mu$ by the relations

\[(s + 1)^p = s + 1 + \lambda p, \quad sp^2 = s + \mu p.\]

Then

\[(2) \quad (s + 1)^p = sp^2 + 1 + (\lambda - \mu)p.\]

We have also

\[(s + 1)^p \equiv (s + 1)^p + \lambda p^2(s + 1)^p-1 \mod p^3\]

\[= s + 1 + \lambda p + \lambda p^2 \mod p^3\]

\[= s + 1 + \lambda (p + p^2) \mod p^3.\]

Likewise

\[sp^2 \equiv s + \mu(p + p^2) \mod p^3.\]

From the last two congruences we have

\[(3) \quad (s + 1)^p \equiv sp^2 + 1 + (\lambda - \mu)(p + p^2) \mod p^3.\]

From (2) and (3) we see that a necessary and sufficient condition for either (1) or (1') is that $\lambda - \mu \equiv 0 \mod p^2$. Therefore (1) and (1') are equivalent.

The simpler relation (1') can be derived more readily than the relation (1). For from the congruence $x + y + z \equiv 0 \mod p^2$, obtained in my previous paper, we have immediately $(x + y)^p \equiv -z^p \mod p^3$. Hence
\[(x + y)^p = x^p + y^p \mod p^3,\]
from which \((1')\) is readily deduced.

Professor Birkhoff points out further that the test fails to be effective for all primes \(p\) of the form \(6n + 1\). For if \(p = 6n + 1\) it follows from the theory of primitive roots modulo \(p^3\) that the congruence
\[t^3 \equiv 1 \mod p^3\]
has a solution \(t\) for which \(t - 1\) is prime to \(p\). Hence also
\[t^2 + t + 1 \equiv 0 \mod p^3.\]
Then we have
\[(t + 1)^p = (t + 1)(t + 1)^{6n} \equiv (t + 1)(-\ell^3)^{6n} \equiv t + 1 \mod p^3,
(t + 1)^{p^2} = (t + 1)^p = t + 1 \mod p^3,
\]
and
\[t^p \equiv t \cdot t^{6n} \equiv t \mod p^3, \quad p^2 \equiv t^p \equiv t \mod p^3.\]
Therefore
\[(t + 1)^{p^2} \equiv t^{p^2} + 1 \mod p^3.\]
Now put
\[t = \sigma + \ell p, \quad (0 < \sigma < p - 1).\]
Then
\[t^{p^2} \equiv \sigma^{p^2}, \quad (t + 1)^{p^2} \equiv (\sigma + 1)^{p^2} \mod p^3.\]
Therefore
\[(\sigma + 1)^{p^2} \equiv \sigma^{p^2} + 1 \mod p^3, \quad (0 < \sigma < p - 1).\]
This is relation \((7)\) of my previous note; from this follows \((1)\) as in the earlier treatment. Hence \((1)\) is satisfied by all primes of the form \(6n + 1\). Therefore the test can be useful only when the exponent \(p\) is 3 or is of the form \(6n - 1\).

\textbf{AN EXTENSION OF A THEOREM OF PAINLEVÉ.}

\textbf{BY DR. E. H. TAYLOR.}

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\textbf{Theorem:} Let \(f(z)\) be a function which is single-valued and analytic throughout the interior of a region \(S\) of the \(z\)-plane, \(z = x + yi\). If \(f(z)\) vanishes at every point of a