that this limit for u cannot be much increased. But if a limit of simpler form is desired one may use

$$u > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{n} - 1.$$

If n is greater than 22, this number is less than $\frac{1}{4}n(1 + \sqrt{1 - 4/\sqrt{n}})$; in fact the latter approaches $\frac{1}{2}n - \frac{1}{2}\sqrt{n} - \frac{1}{2}$ as n increases. If n is less than 23 it is known that this limit holds, for all primitive groups of class less than 14 are known,* and the classes 4 and 6, which alone are in question here, belong to no primitive groups of higher degree than 10.

STANFORD UNIVERSITY,
October, 1913.

CHRISTOFFEL'S MATHEMATICAL WORKS.

E. B. Christoffel, Gesammelte mathematische Abhandlungen, Unter Mitwirkung von A. KRAZER und G. FABER, herausgegeben von L. MAURER. Zwei Bände. B. G. Teubner, Leipzig, 1910.

WHEN one turns over the pages of the collected works of a mathematician such as this one, arranged in chronological order, and notes the varied fields in which the author worked, he feels an impulse to follow the methods of his literary colleagues and to try to find the influences which played upon the author. To what extent was he influenced by direct contact with other masters? Or perhaps was he that year lecturing upon a certain subject and thus was naturally led to an attempt to solve some of its problems? These and other questions arise in the mind of a reviewer, and he must decide whether he shall amuse himself chasing fancies or turn to the more serious task of the kind of a review such as we are accustomed to expect.

In the present instance some of the former questions receive a partial answer as he reads the interesting biography of the author written by Dr. C. F. Geiser for the thirty-fourth volume of the *Mathematische Annalen* and reprinted at the beginning of the first volume now under discussion. Here one reads that in his student days at Berlin Christoffel came under the influence of Dirichlet, Borchardt, and Steiner, and later,

* Cf. Manning, *Amer. Jour. of Math.*, vol. 35 (1913), p. 229, for references.

in his Docent days, of Weierstrass and Kummer. At the same time he studied under Dove and Ohm, which may account for his early tendency in the direction of mathematical physics, but the influence of the great masters of pure mathematics is seen in his writings on these subjects as well. For example his memoir, "Sul problema delle temperature stazionarie e la rappresentazione di una superficie," deals essentially with the conformal representation of plane areas, and as he says, although he retains the problem in the form given by physics, his principal interest is in its bearing on functions of a complex variable. Some of his results suggest the classic work of Schwarz in this field. His biographer gives one a fine estimate of Christoffel's power as a lecturer and of his personal influence on his students and colleagues.

The first volume contains his thesis and eighteen papers written for the most part while he was professor in the Polytechnikum in Zürich. They are upon a variety of topics, but it will be possible for us to refer to only a few of them. One of the most extensive is entitled "Zur Theorie der einwerthigen Potentiale." Herein Christoffel has investigated the general properties of functions which are expressible as the sum of the four types of potential, namely point, line, surface, and solid. In the first three memoirs of the second volume he solves the problem of finding the most general form of functions which effect a conformal mapping upon a circle of a simply-connected plane surface of n-sides, of which Schwarz also gave the solution. Quite a number of the other memoirs deal with one phase or another of the theory of the potential.

The title "Ueber einige allgemeine Eigenschaften der Minimumsflächen" applied to Memoir XIII is misleading because, although it deals with a property possessed by minimal surfaces, it is concerned with the solution of the problem: Given two real surfaces S and S' corresponding with parallelism of tangent planes; under what conditions is this correspondence conformal? It is shown first that the lines of curvature on the two surfaces must have the same spherical representation and that the principal radii of curvature of the two surfaces must satisfy the condition $\rho_1\rho_2' + \rho_2\rho_1' = 0$. Later Christoffel finds that both S and S' are isothermic surfaces; and further that given any isothermic surface, not a sphere, there exists a second surface S' in the above relation, and it can be found by quadratures. This is the so-called transformation of

Christoffel of isothermic surfaces. In 1899 Darboux* inquired under what conditions the correspondence between the two sheets of the envelope of a two-parameter family of spheres is conformal, and found that, with an evident exception, both surfaces must be isothermic. Bianchi called these transformations D_m and showed that they are commutative with the transformation of Christoffel. Recently Demoulin† established the existence of a generalized transformation of Christoffel of isothermic orthogonal reseaux in n-space and proved that the transformations D_m correspond to the case where $n = 4$.

In the memoir "Allegemeine Theorie der geodätischen Dreiecke" one finds for the first time the so-called *Christoffel symbols* $\left\{ \begin{matrix} rs \\ \hline t \end{matrix} \right\}$, which denote certain functions of the coefficients of a quadratic differential form and their first derivatives. It is of interest that these symbols are introduced in connection with the binary quadratic differential form which is the linear element of the surface whose geodesics are under discussion, although they are used subsequently in connection with the investigation of general quadratic differential forms. A large part of this paper is given over to the theory of the *reduced length of a geodesic segment*, a geometrical entity of value in the discussion of geodesics near a given geodesic. This may be defined as follows: If M_0M is a segment of a geodesic and if one draws through M_0 a geodesic making an infinitesimal angle θ with M_0M and through M an arc MH perpendicular to the first geodesic and meeting the second in H, then MH/θ is the reduced length of MM_0, and is written $[M_0M]$. Christoffel's opinion of this element is given in the words "I consider it to be the principal result of the following investigations that, just as the theory of forces of attraction working in accordance with Newton's law depends upon a single function—the potential—so the geodesy of an arbitrary curved surface comes back to the theory of a single function of four variables, which I call the reduced length of a geodesic segment, and which I denote by $[OO']$ where O and O' are the end points of the segment."

The last chapter is devoted to a *geodesic classification* of

* "Sur les surfaces isothermiques," *Annales de l'École Normale Supérieure*, ser. 3, vol. 16 (1899), p. 498.

† *Comptes Rendus*, vol. 153 (1911), p. 927.

curved surfaces. According as there are no relations between the sides and angles of a geodesic triangle on a surface, or one, two or three relations, the surface is said to be the first, second, third or fourth class. Christoffel showed that every surface of constant curvature belongs to the fourth class, but he was unable to arrive at any other essential results in the classification. However, Weingarten* showed by a different method that there are no surfaces of the third class and that all of the fourth class have constant curvature. Later, H. von Mangoldt† proved that the second class contains surfaces of revolution of variable curvature and surfaces applicable to them.

The last three papers of the first volume contain Christoffel's important contribution to the theory of invariants of quadratic differential forms. The fundamental problem may be stated as follows:

Given two differential quadratic forms in n variables

$$F = \sum_{i,k} a_{ik} dx_i dx_k, \quad F' = \sum_{i,k} a_{ik}' dy_i dy_k, \tag{1}$$

where a_{ik} and a'_{ik} are functions of the x's and y's respectively; what are the necessary and sufficient conditions that a transformation of the variables exist, say

$$x_i = \varphi_i(y_1, \cdots, y_n), \tag{2}$$

such that F is transformable into F'?

This is equivalent to the determination of the conditions under which the system of partial differential equations

$$a_{ik}' = \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial y_i} \frac{\partial x_s}{\partial y_k} \tag{3}$$

admits a solution. This question is reduced by Christoffel to an algebraic problem.

If equations (3) be differentiated with respect to y_l, where $l = 1, \cdots, n$, and the resulting equations be solved for the second derivatives, the conditions that these equations be consistent are expressible in the form

$$(\alpha_1\alpha_2\alpha_3\alpha_4)' = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \frac{\partial x_{i_2}}{\partial y_{\alpha_2}} \frac{\partial x_{i_3}}{\partial y_{\alpha_3}} \frac{\partial x_{i_4}}{\partial y_{\alpha_4}}, \tag{4}$$

* "Ueber die Verschiebbarkeit geodätischer Dreiecke in krummen Flächen," *Sitzungsberichte der K. P. Akademie der Wissenschaften zu Berlin*, 1882, p. 453.

† "Ueber die Classification der Flächen nach der Verschiebbarkeit ihrer geodätischen Dreiecke," *Crelle*, vol. 94 (1882), p. 21.

where $(i_1i_2i_3i_4)$ denotes a certain function of the quantities a_{rs} and their first and second derivatives, and $(\alpha_1\alpha_2\alpha_3\alpha_4)'$ the same function in terms of the functions a_{rs}'. These functions $(i_1i_2i_3i_4)$ were used first by Riemann in the Commentatio Mathematica and so Ricci has called them the *Riemann symbols*.

Similarly to (3), equations (4) are the conditions of equivalence of a form

$$G_4 = \sum_{i_1i_2i_3i_4} (i_1i_2i_3i_4) d^{(1)}x_{i_1} d^{(2)}x_{i_2} d^{(3)}x_{i_3} d^{(4)}x_{i_4}, \tag{5}$$

and an analogous form G'_4 in the y's. Here we take four different sets of differentials, since there are certain linear relations between Riemann symbols, in consequence of which G_4 vanishes identically when the differentials are the same. Thus equations (4) are the conditions for the equivalence of two quadrilinear forms.

If equations (4) be differentiated with respect to y_α and if we make use of the five index symbols defined by

$$(i_1i_2i_3i_4i) = \frac{\partial}{\partial x_i}(i_1i_2i_3i_4) - \sum_\lambda \left[\begin{Bmatrix} ii_1 \\ \lambda \end{Bmatrix} (\lambda i_2i_3i_4) + \begin{Bmatrix} ii_2 \\ \lambda \end{Bmatrix} (i_1\lambda i_3i_4) + \cdots \right], \tag{6}$$

where $\begin{Bmatrix} ij \\ k \end{Bmatrix}$ is the usual Christoffel symbol, we obtain equations

$$(\alpha_1\alpha_2\alpha_3\alpha_4\alpha)' = \sum_{i_1i_2i_3i_4i} (i_1i_2i_3i_4i) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \cdots \frac{\partial x_i}{\partial y_\alpha}, \tag{7}$$

which evidently are the conditions for the equivalence of two quinquilinear forms G_5 and G_5'.

Continuing by means of a process which is an immediate generalization of (6), we obtain a series of equations analogous to (7) and thus a sequence of covariant forms $G_4, G_5, \cdots, G_\mu$.

These results are developed at length in Memoir XVIII and the properties of the Riemann symbols are investigated. Here one finds for the first time, the so-called Christoffel symbols of the first kind.

After these preliminaries the author establishes the following fundamental theorem:

The necessary and sufficient conditions in order that it shall be possible to transform a quadratic form F into another quadratic form F' are that the equations in the variables x, y, $\partial x/\partial y$ derived from the equivalence of two sequences

$$F,\ G_4,\ G_5,\ \cdots,\ G_\mu \text{ and } F',\ G_4',\ G_5',\ \cdots,\ G_\mu'$$

shall be algebraically compatible.

It is necessary to distinguish between the following two cases:

(i) When an order q can be determined such that the equations

$$F = F',\ G_4 = G_4',\ \cdots,\ G_{q-1} = G_{q-1}'$$

determine the quantities x and $\partial x/\partial y$ as functions of the y's and these values make $G_q = G_q'$ an identity.

(ii) When the above equations yield only $p(<n^2 + n)$ independent equations, and if any set of solutions of these p equations satisfy identically $G_q = G_q'$.

When the conditions of the first case are satisfied the problem is algebraic, but in the second case certain of the quantities left undetermined by the algebraic conditions must satisfy partial differential equations of the first order. From the algebraic theory it follows that the algebraic invariants of the two sets of multilinear forms must be equal. And the equations of transformation are given by equating n independent absolute invariants. Hence it is necessary that the sequence of forms F, G_4, G_5, G_6, $\cdots$ be extended to such a point that there shall be n absolute simultaneous invariants. In particular the invariants I, I_1, $\cdots$ of the algebraic forms F, G_4, $\cdots$ are a complete system of relative differential invariants of the form F.

It was the methods followed in these memoirs which served as the foundation for the interesting *Calcul differential absolu* which has been developed by Ricci and Levi-Civita.* They call the operation defined by equation (6) *covariant differentiation*. It plays a fundamental rôle in this method which has not received the consideration it deserves.

During the latter years of his life Christoffel's researches dealt primarily with algebraic functions and their integrals, a subject which retains its interest at present. As representative of his important contributions to this field, one may mention his extensive memoirs entitled "Ueber die canonische Form der

* "Méthodes de calcul différentiel absolu et leurs applications," *Math. Annalen*, vol. 54, pp. 128–196.

Riemannschen Integrale erster Gattung" and "Algebraischer Beweis des Satzes von der Anzahl der linearunabhängigen Integrale erster Gattung," the titles of which convey a sufficient idea of the problems solved. However, it is a question whether Christoffel's later work was as novel and fundamental as that which we have discussed at greater length above.

At the end of certain memoirs the editor has added remarks which are helpful. In every way the books present a fine appearance, which may be a superfluous observation since they are published by Teubner.

LUTHER PFAHLER EISENHART.

PROBLEM COLLECTIONS IN CALCULUS.

Esercizi di Analisi infinitesimale. Di G. VIVANTI. Puntata I. Completo. Mattei, Pavia, 1912. ix + 470 pp. Price, 15 lire.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Von F. DINGELDEY. (Lehrbücher der Mathematischen Wissenschaften $\text{XXXII}_{1,2}$.) I. Teil; Differentialrechnung, 1910, vi + 202 pp.; II. Teil; Integralrechnung, 1913, ii + 382 pp. Teubner, Berlin und Leipzig. Price, 6 + 13 marks.

MOST treatises on the calculus contain numerous solved and unsolved problems, but in what follows I wish to indicate some of the more notable separately published problem collections which, chiefly by reason of the industry of Germans, present a most formidable array before an inquirer. There are, on the one hand, works which simply contain problems to solve, as those of Byerly* and Wolstenholme.† On the other hand, we have the voluminous collection of books which give a synopsis of a certain amount of theory, set forth numerous worked out examples of a somewhat typical nature, and give similar problems for solution. This is the style of the little works by

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* W. E. Byerly, Problems in Differential Calculus supplementary to a Treatise on Differential Calculus. Boston, 1895, pp. viii+71.

† J. Wolstenholme, Mathematical Problems on the Subjects for the Cambridge Math. Tripos Examination, Part I, 3d ed., Lond., 1891. Problems 1641–1992 on Differential Calculus, Higher Plane Curves, Integral Calculus.