ON A GENERALIZATION OF A THEOREM OF DINI ON SEQUENCES OF CONTINUOUS FUNCTIONS.

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We propose in this note to give a generalization of the following theorem of Dini*: "If a monotonic sequence of functions continuous on a closed interval converges to a continuous function, the convergence is uniform."

The double sequence analogue of this theorem proves to be of importance in our generalization. We embody it in the following

**Lemma.** If a double sequence $a_{mn}$ is monotonic non-decreasing in $m$ for every $n$, and if $L_m L_n a_{mn} = L_n L_m a_{mn}$, all the limits being supposed to exist, then $L_m a_{mn}$ and $L_n a_{mn}$ converge uniformly and the double limit $L_{mn} a_{mn}$ exists and is equal to the iterated limits.

The proof of the uniformity of convergence of $L_m a_{mn}$ is part of Theorem I of the paper by the author in the *Annals of Mathematics*, series 2, volume 14, page 81. This uniformity has as a direct consequence the existence of the double limit equal to the iterated limits, which in turn implies the uniformity of $L_n a_{mn}$.

For the purposes of generalization, consider a class $\mathcal{D}$ of elements unconditioned excepting for the existence within the class of some definition of limit, i.e., some means of determining whether a sequence of elements has a limit and what this limit is.† Then it is possible to define the concepts limiting element, closed and compact relative to subclasses $R$ of $\mathcal{D}$.‡ Also if $\mu$ is a real-valued function on $R$, we can define the notion of continuity, as well as equal continuity, as applied to a set of functions. In such a situation we are able to state the following

**Theorem.** If $R$ is a closed and compact subclass of $\mathcal{D}$, and $\mu_{nr}$ is a monotonic sequence of functions continuous on $R$ and

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converging to the continuous function \( \mu_r \), then the convergence is uniform and the functions \( \mu_{nr} \) are equally continuous.

We prove the uniformity of convergence of \( \mu_{nr} \). Since \( L_n \mu_{nr} = \mu_r \) for every \( r \) of the class \( \mathcal{R} \), we have: for every positive \( \varepsilon \) there exists an \( n_{\varepsilon r} \) such that if \( n \geq n_{\varepsilon r} \) we have \( |\mu_{nr} - \mu_r| \leq \varepsilon \). Suppose that we have selected for each \( \varepsilon \) and \( r \) the smallest possible value as \( n_{\varepsilon r} \). Then we wish to show that for each \( \varepsilon, \) \( n_{\varepsilon r} \) is bounded on the class \( \mathcal{R} \). Suppose, if possible, this were not so for some particular \( \varepsilon \). Then for every \( n \), there would exist an \( r_n \) such that \( n > n_{\varepsilon r} \), i. e., \( |\mu_{nr_n} - \mu_r| > \varepsilon \). On account of the convergence of \( \mu_{nr} \) to \( \mu_r \) for every \( r \), no element can recur infinitely often in the set \( r_n \).

Then since the class \( \mathcal{R} \) is compact and closed, there will exist a subsequence \( r_{n_m} = r'_m \) of \( r_n \), and an \( r \), such that \( L_m r'_m = r \).

Consider the double sequence \( \mu_{n_m r'_m} \). It is monotonic non-decreasing in \( n \) for every \( m \). Moreover on account of the continuity of \( \mu_{nr} \) and \( \mu_r \), we have \( L_m L_n \mu_{n_m r'_m} = L_n L_m \mu_{n_m r'_m} \). It therefore fulfils the conditions of our lemma, and it follows that \( L_n \mu_{n_m r'_m} \) converges uniformly; i. e., for every positive \( \varepsilon \) there will exist an \( n_{\varepsilon m} \), independent of \( m \), such that \( |\mu_{n_m r'_m} - \mu_r| \leq \varepsilon \). By taking the \( \varepsilon \) as the one presupposed above, and \( n > n_{\varepsilon m} \) we obtain a contradiction.

The equal continuity of the functions \( \mu_{nr} \) is a direct consequence of their uniformity of convergence and continuity.

To obtain a further generalization we presuppose another general class \( \mathcal{B} \). In \( \mathcal{B} \) we shall suppose that there is defined an order relation between triplets of elements: \( B_p q q r \), comparable to \( p \leq q \leq r \). We shall suppose that there exists in the class also the concept of limit, subject to the condition that, if \( L_n p_n = p \), then there exists a subsequence having the same limit, such that \( B_{p_n q_n r} \) for every \( m \), or \( B_{p_n q_n r} \) for every \( m \). If \( \mu \) is a real valued function on \( \mathcal{B} \), a subclass of \( \mathcal{B} \), then \( \mu \) is said to be monotonic non-decreasing on \( \mathcal{B} \) if for every triplet \( s_1, s_2, s_3 \) of \( \mathcal{B} \) such that \( B_{s_1 s_2 s_3} \) we have \( \mu_{s_1} \leq \mu_{s_2} \leq \mu_{s_3} \). Finally in the composite class \( \mathcal{B} \), we obtain a double limit, viz., \( L_m p_m q_n = pq \) is equivalent to \( L_m p_m = p \) and \( L_n q_n = q \). This enables us to define a continuity of functions on a composite range, similar to that of continuity of functions of two variables, viz., \( \mu \) is continuous on \( \mathcal{B} \) if \( L_m s m r_n = sr \) implies \( L_m \mu_{s m r_n} = \mu_{sr} \). Then we have the following

Theorem. If $\mathcal{S}$ and $\mathcal{R}$ are closed and compact subclasses of $\mathcal{P}$ and $\mathcal{Q}$, respectively, if further $\mu_{sr}$ is continuous on $\mathcal{S}$ for every $r$ and on $\mathcal{R}$ for every $s$, and if moreover $\mu_{sr}$ is monotone non-decreasing on $\mathcal{S}$ for every $r$, then $\mu_{sr}$, considered as a set of functions on $\mathcal{R}$, are equally continuous, as well as $\mu_{sr}$ considered as a set of functions on $\mathcal{S}$, and $\mu_{sr}$ is continuous on $\mathcal{SR}$.

The proof that $\mu_{sr}$, considered as a set of functions on $\mathcal{R}$, are equally continuous is an indirect one. The assumption that $\mu_{sr}$ is not equally continuous on $\mathcal{R}$ is shown to be untenable by a use of the property of limit in terms of $B$, the monotoneity and continuity of $\mu_{sr}$, and the preceding theorem. The details are easily supplied. The equal continuity of the set $\mu_{sr}$ on $\mathcal{S}$, and the continuity on $\mathcal{SR}$ follow at once from the equal continuity on $\mathcal{R}$.

By specializing the classes $\mathcal{P}$ and $\mathcal{Q}$, we get some interesting theorems in special fields. If we take $\mathcal{P} = 1, 2, 3, \ldots, \infty$, with $B_{p_1p_2p_3}$ defined as $p_1 \leq p_2 \leq p_3$, and $\mathcal{Q}$ as the interval $0 \leq x \leq 1$, and note that equal continuity on $1, 2, 3, \ldots, \infty$ is uniform convergence, we get the Dini theorem stated at the outset. If $\mathcal{P}$ is the linear interval $0 \leq x \leq 1$, and $B_{p_1p_2p_3}$ is the same as $p_1 \leq p_2 \leq p_3$, and $\mathcal{Q}$ is the set $1, 2, 3, \ldots, \infty$, we have:

If a sequence of monotone non-decreasing functions continuous on a closed interval converges to a continuous function, the convergence is uniform, and the set of functions are equally continuous.*

If $\mathcal{P}$ is the linear interval $0 \leq x \leq 1$, with $B_{p_1p_2p_3}$ equivalent to $p_1 \leq p_2 \leq p_3$ and $\mathcal{Q}$ is the linear interval $0 \leq y \leq 1$, we have:

If $f(x, y)$, defined for $0 \leq x \leq 1$, $0 \leq y \leq 1$, is continuous in $x$ for every $y$, and in $y$ for every $x$, and is also monotone non-decreasing in $x$ for every $y$, then $f(x, y)$ is continuous in $x$ and $y$ simultaneously.

* Cf. Buchanan and Hildebrandt: *Annals of Mathematics*, ser. 2, vol. 9, p. 123. It is interesting to observe that this theorem and the Dini theorem are special cases of the same theorem.