of ω_u^2 is seen to be

$$H_{x_ux_u}X^2 + 2H_{x_uy_u}XY + H_{y_uy_u}Y^2 + 2H_{x_uz_u}XZ + 2H_{y_uz_u}YZ + H_{z_uz_u}Z^2.$$

Equations (12), with F replaced by H, reduce this to the form

$$H_{11}(X^2 + Y^2 + Z^2)^2 = H_{11}.$$

Similarly the coefficients of $\omega_u\omega_v$ and ω_v^2 can be proved equal to $2H_{12}$ and H_{22} respectively. The other coefficients will be called H_{00}, $2H_{01}$ and $2H_{02}$ respectively, and equation (11) becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (H_{00}\omega^2 + 2H_{01}\omega\omega_u + 2H_{02}\omega\omega_v + H_{11}\omega_u^2 + 2H_{12}\omega_u\omega_v + H_{22}\omega_v^2)dudv.$$

This equation is in the same form as equation (5), and from this point on the argument is so nearly the same as in the non-parametric case that it need not be repeated here. The analogue of inequality (10) is seen to be

$$H_{11}(x, y, z, x_u, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, \cdots, z_v; \lambda) \geqq 0.$$

COLUMBIA UNIVERSITY.

NOTE ON THE DERIVATIVE AND THE VARIATION OF A FUNCTION DEPENDING ON ALL THE VALUES OF ANOTHER FUNCTION.

BY PROFESSOR G. C. EVANS.

(Read before the American Mathematical Society, January 2, 1915.)

1. IN a recent article,* Fréchet has given a treatment of the differential of a function depending on a curve, by making use of and evaluating Riesz's expression of a linear relation in terms of a Stieltjes integral. According to Fréchet, if $\underset{a}{\overset{b}{F}}[\varphi]$ depends on all the values of $\varphi(x)$ between a and b, then

* M. Fréchet, "Sur la notion de différentielle d'une fonction de ligne," *Transactions of the American Mathematical Society*, vol. 15 (1914), pp. 135–161.

if the differential of F exists it is given by the formula

$$dF[\varphi(x)] = \int_a^b \Delta\varphi(\xi) d\alpha_\phi(\xi), \tag{1}$$

where $\Delta\varphi(\xi)$ is the increment in φ, and $\alpha_\phi(\xi)$ is some function of finite variation. On the other hand, Volterra* has shown that, under certain conditions, the variation of F is given by the formula

$$\delta F[\varphi(x)] = \int_a^b F'[\varphi(x)|\xi]\delta\varphi(\xi)d\xi, \tag{2}$$

where $F'\varphi[(x) \mid \xi]$ is the functional derivative of F with respect to $\varphi(x)$ at the point ξ. The integral in (1), however, according to Fréchet, itself splits up into three parts, of which one has a form similar to (2), so that as a special case we should have

$$dF[\varphi(x)] = \int_a^b A[\varphi(x) \mid \xi]\Delta\varphi(\xi)d\xi. \tag{3}$$

It is the object of this short paper, in the first place, to derive the formula (2) under slightly less restrictive conditions than those of Volterra, and in the second place, by adopting a point of view more akin to that of Fréchet, to show the relation between equation (1) and equations (2) and (3).

2. We shall consider as a region for the argument $\varphi(x)$ that included between two given continuous functions $\Phi_1(x)$ and $\Phi_2(x)$, where $\Phi_1(x) < \Phi_2(x)$, in the interval $a \leqq x \leqq b$; i. e., the region

$$\Phi_1(x) \leqq \varphi(x) \leqq \Phi_2(x), \qquad a \leqq x \leqq b,$$

and we shall assume that $F[\varphi]$ is defined for every continuous function in that region, and is continuous.† This we shall call the assumption (α).

In addition to (α), in order to obtain formula (2), Volterra makes four assumptions I–IV. By a different method of proof, however,—the one which we first adopt—it is possible to arrive at (2) by means of (α), (II) and (III) alone, from which

* V. Volterra, "Sopra le funzioni che dipendono da altre funzioni," *Rendiconti della R. Accademia dei Lincei*, vol. 3 (1887), pp. 97–105, 141–146, 153–158.

† We mean that $F[\varphi]$ has continuity of the zeroth order.

IV follows, but not (I) in its entirety.* These assumptions are as follows.

(II) Take an interval h within ab and give to $\varphi(x)$ within h a continuous variation $\theta(x)$, of one sign, such that $|\theta(x)| < \epsilon$. Denote by ΔF the corresponding change of F, and write $\sigma = \int_h \theta(x)dx$. It is assumed that the ratio $\Delta F/\sigma$ approaches a fixed limit (denoted by $F'[\varphi(x) \mid \xi]$ and spoken of as the functional derivative of F) as ϵ and h approach zero in an arbitrary manner, provided that the interval h always contains within itself the value $x = \xi$.†

(III) It is assumed that the ratio $\Delta F/\sigma$ approaches its limit uniformly with respect to all possible functions $\varphi(x)$ and values ξ.

Let us for convenience denote by (III_1) that part of (III) which requires uniformity with respect to the functional argument alone.

I. First Deduction of Formula (2).

3. The following theorem is introductory.

THEOREM 1. *If for a certain continuous function $\varphi_0(x)$ the function $F[\varphi(x)]$, continuous in $\varphi(x)$, has a derivative $F'[\varphi(x) \mid \xi]$ for every value of ξ in a certain closed sub-interval $a'b'$ of ab,‡ that derivative remains finite and continuous throughout $a'b'$.*

To prove this theorem, let $\xi_1, \xi_2, \cdots$ be any infinite set of values of ξ in $a'b'$ having ξ_0 as a limiting point, and suppose that

$$F'[\varphi_0(x) \mid \xi_0] = t, \qquad \lim_{n=\infty} F'[\varphi_0(x) \mid \xi_n] = t'.$$

We shall assume that t' is finite; the case where t' is infinite occasions an obvious modification of the proof. Let $|t-t'|=p$, and suppose momentarily $p \neq 0$.

Give to $\varphi_0(x)$ a variation of one sign, $\theta_1(x)$, of the kind specified in (II), and take ϵ_1 and h_1 so small that we have the

* See § 4, and footnote.

† We understand here that $\theta(\xi) \neq 0$. This restriction turns out to be immaterial, but makes the definition correspond more closely to that of the ordinary derivative, where a restriction somewhat related to this is essential to the nature of the operation.

This definition requires obvious modification when $\xi = a$ or b and when $\varphi(x) = \Phi_1(x)$ or $\Phi_2(x)$.

‡ In particular, we may take $a' = a$ and $b' = b$.

inequality

$$(4) \qquad | F[\varphi_0(x) + \theta_1(x)] - F[(\varphi_0(x)] - t\sigma_1 | < \left| \frac{1}{2} p\sigma_1 \right|.$$

We may, however, by taking a variation θ_2, small enough, and about a point ξ_n near enough to ξ_0, and adding to it a variation θ_3, small enough everywhere, yet different from zero at ξ_0, obtain a variation θ_1 for which is satisfied the inequality

$$(5) \qquad \begin{aligned} &| F[\varphi_0(x) + \theta_2(x) + \theta_3(x)] - F[\varphi_0(x)] \\ &\qquad\qquad - t'(\sigma_2 + \sigma_3) | < \left| \frac{1}{2}\, p(\sigma_2 + \sigma_3) \right|. \end{aligned}$$

For, since F is assumed to be continuous, the increment θ_3 may be made so small as to affect the difference $F[\varphi_0 + \theta_2] - F[\varphi_0]$ by as little as we please. But from (4) and (5), by taking $\theta_1 = \theta_2 + \theta_3$, it follows that

$$| (t - t')(\sigma_2 + \sigma_3) | < p \,|\, \sigma_2 + \sigma_3 \,|,$$

which is a contradiction. Hence $p = 0$, and the theorem is proved.

4. Let us now make the assumptions (α), (II), (III_1). It follows at once that *the derivative* $F'[\varphi(x) \mid \xi]$ *is continuous in regard to* $\varphi(x)$, *if* ξ *is any fixed value in the interval* ab. For we have

$$\left| F'[\varphi(x) \mid \xi] - \frac{F[\varphi(x) + \theta(x)] - F[\varphi(x)]}{\sigma} \right| < \eta,$$

and on account of the condition of uniformity,

$$\begin{aligned} &| F'[\varphi_2(x) \mid \xi] - F'[\varphi_1(x) \mid \xi] | \\ &\qquad\qquad \leqq | 1/\sigma \{F[\varphi_2(x) + \theta(x)] - F[\varphi_1(x) + \theta(x)]\} | + 2\eta. \end{aligned}$$

Hence first fixing the ϵ and h corresponding to $\theta(x)$ small enough so that 2η is less, say, than $\omega/2$, we can then take φ_2 near enough to φ_1, the $\theta(x)$ being fixed, so that the other part of the expression is also less than $\omega/2$. That is to say, by taking $| \varphi_2(x) - \varphi_1(x) |$ small enough, we can make the left hand member of the inequality as small as we please. Hence F' has continuity of the zeroth order with respect to φ.

From this it follows that F' is continuous uniformly with respect to φ, if φ is restricted to any family of curves whose ordinates are uniformly continuous functions of a finite number of parameters, over a perfect domain for those parameters, or if φ is restricted to a family of curves closed in the sense that the limiting curves are uniform limits; but not that F' is continuous in φ uniformly with respect to all continuous functions φ in the given region, nor even that F' is continuous in ξ uniformly for all continuous functions φ in the given region. *If however we make the whole of assumption* (III), it follows, as is easily verifiable, that *F' is continuous in ξ uniformly for all points ξ and all continuous curves φ in the region,* and hence that *F' is continuous in ξ and φ uniformly with respect to ξ and φ, provided that φ is restricted to a family of continuous functions, closed in the sense that the limiting functions are uniform limits.**

5. We can now prove, by means of the assumptions (α), (II), (III_1), something analogous to Rolle's theorem.

THEOREM 2. *Let $F[\varphi(x)]$ be a function for which* (α), (II), (III_1) *hold, and let $F[\varphi_1] = F[\varphi_2] = 0$, where $\varphi_1 - \varphi_2$ is a function which does not change sign in the interval ab, and is different from zero only in the interval $a'b'$. Then there is a function φ_0, of the pencil determined by φ_1 and φ_2, and a value $\xi_0 (a' \leqq \xi_0 \leqq b')$, such that $F'[\varphi_0(x) \mid \xi_0] = 0$.*

In fact, if we write $F[\varphi_1 + \omega(\varphi_2 - \varphi_1)]$ as a function of ω, $F(\omega)$, it will be continuous in ω, and for a certain value $\omega = \omega_0$ will attain its maximum or minimum. Let this value of ω determine the function φ_0, and for the sake of definiteness, let us assume that $F(\omega) - F(\omega_0)$ is not positive if ω is in the neighborhood of ω_0. For the sake of definiteness also, let us assume that the functional derivative of F is positive when $\xi = b'$. Then it must be positive throughout the whole of the closed interval $a'b'$, unless it vanishes at some point of that interval, since it is continuous in ξ. Let us assume that it does not vanish.

In order to show the falsity of this assumption, let us construct the functions

$$
\begin{aligned}
\psi_{\omega,t}(x) &= 0 && (a \leqq x \leqq t,\ b' \leqq x \leqq b) \\
(6) \qquad &= (x - t)\{\varphi_2(x) - \varphi_1(x)\} && (t \leqq x \leqq t + \omega) \\
&= \omega\{\varphi_2(x) - \varphi_1(x)\} && (t + \omega \leqq x \leqq b'),
\end{aligned}
$$

* This is essentially what Volterra uses as his postulate (IV). See loc. cit., p. 99 and p. 101. His postulate (I) is that if $\theta(x)$ is any variation of φ in h (not necessarily of one sign) in absolute value less than ϵ, then $|\Delta F/\epsilon h| < M$, uniformly.

and consider for what value of t, $t = \bar{t}$, the difference

$$F[\varphi_0(x) + \psi_{\omega,\,t}(x)] - F[\varphi_0(x)]$$

attains its upper limit, if ω is kept constant. We know that there will be such a value, since F is a continuous function of t, and we see directly that we can find ω' small enough so that if $\omega \leqq \omega'$ that value must be greater than a'. For if not, on account of the uniformity of the condition (III_1) and the resulting continuity of the derivative with regard to its functional argument, it would follow that we might take ϵ and h about a' so small that

$$F[\varphi(x) + \theta(x)] - F[\varphi(x)] > 0,$$

where $\theta(x)$ is any variation in the interval h, everywhere in absolute value less than ϵ,* and $\varphi(x)$ is any one of the functions $\varphi(x) = \varphi_0(x) + \psi_{\omega,\,t}(x)$. But this means that we could take ω small enough so that we should have $F[\varphi_0 + \omega(\varphi_2 - \varphi_1)] - F[\varphi_0] > 0$, which would be contrary to hypothesis.

Let us now take a series of values ω_n which approach zero as a limit. The corresponding series of values t_n of $\bar{t}$ has at least one limiting value, and any one of these limiting values, which may in particular be the point a', we may take as our ξ_0. This gives us our contradiction. For if $F'[\varphi_0(x) \mid \xi_0]$ were not zero, we could, owing to the uniformity of the condition (III_1), construct a function $\psi_{\omega_n,\,t}(x)$, with $t < t_n$, for which we should have

$$F[\varphi + \psi_{\omega_n,\,t}] > F[\varphi + \psi_{\omega_n,\,t_n}].$$

Our theorem is therefore proved.

6. The law of the mean is a consequence of this theorem in the same way as in differential calculus it is a consequence of Rolle's theorem.

LAW OF THE MEAN.† *Let $F[\varphi(x)]$ be a function for which* (α), (II), (III_1) *hold, and let φ_1 and φ_2 be two continuous functions in the given region, such that $\varphi_1 - \varphi_2$ does not change sign in the interval ab, and is different from zero only in the interval $a'b'$.*

* If $a' = a$, it is taken for granted that $\varphi_2(a) - \varphi_1(a)$ and $\theta(a)$ need not necessarily vanish. Also if $b' = b$, it is assumed that $\varphi_2(b) - \varphi_1(b)$ need not necessarily vanish.

† This theorem is due to Volterra (loc. cit., p. 103) who establishes it by means of formula (2), and thus as a consequence of the hypotheses (α) (I)–(IV).

Then there is a function φ_0 of the pencil determined by φ_1 and φ_2, and a value ξ_0 $(a' \leqq \xi_0 \leqq b')$ such that

$$(7)\qquad F[\varphi_2(x)] - F[\varphi_1(x)] = F'[\varphi_0(x) \mid \xi_0] \int_a^b \{\varphi_2(x) - \varphi_1(x)\} dx.$$

7. We may now proceed, by means of these theorems and the hypotheses (α), (II), (III), to establish the formula (2). Let us consider first a continuous function $\psi(x)$ which does not change sign in the interval ab, and form the function $F(\omega) = F[\varphi_0 + \omega\psi]$. We shall endeavor to calculate $dF/d\omega$ for $\omega = 0$.

If we divide up the interval ab into parts $a_i a_{i+1}$, every one of which is in magnitude less than δ, we can write

$$F[\varphi_0 + \omega\psi] - F[\varphi_0] = \sum_{i=0}^{n-1} F[\varphi_0 + \psi_{\omega,\, a_i}] - F[\varphi_0 + \psi_{\omega,\, a_{i+1}}] \\ + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \omega\psi_{\omega,\, a}],$$

where $a_0 = a$ and $a_n = b$. But this is the same as the expression

$$\omega \sum_{i=0}^{n-1} F'[\varphi_0 + \psi_{\omega_i,\, a_{i+1}} + \gamma_i(\psi_{\omega,\, a_i} - \psi_{\omega_i,\, a_{i+1}}) \mid \xi_i] \\ \times \int_a^b (\psi_{\omega,\, a_i} - \psi_{\omega,\, a_{i+1}}) dx + F[\varphi_0 + \omega\psi] - F[\varphi_0 + \psi_{\omega,\, a}],$$

where $|\gamma_i| < 1$ and $a_i \leqq \xi_i \leqq a_{i+1}$. We may for our purposes take all the intervals equal, and also $\delta = k\omega$ where k is any fixed number greater than 2. If then we take the limit of $\{F[\varphi_0 + \omega\psi] - F[\varphi_0]\}/\omega$ as ω approaches zero, we verify from the uniformity of assumption (III) and the results of § 4 that $dF/d\omega$ exists for $\omega = 0$, and is given by the formula

$$(8)\qquad \left(\frac{dF}{d\omega}\right)_{\omega=0} = \int_a^b F'[\varphi(x) \mid \xi]\psi(\xi) d\xi.$$

Hence it follows that

$$(9)\qquad \left(\frac{dF}{d\omega}\right)_{\omega=\omega} = \int_a^b F'[\varphi(x) + \omega\psi(x)]\psi(\xi) d\xi.$$

8. Consider now the general case where $\psi(x)$ is an arbitrary continuous function. We can write it as $\psi(x) = \psi_1(x) + \psi_2(x)$,

where

$$\psi_1(x) = \tfrac{1}{2}\{\psi(x) + | \psi(x) | \},$$
$$\psi_2(x) = \tfrac{1}{2}\{\psi(x) - | \psi(x) | \},$$

functions which do not change sign. Moreover if we consider the function $F[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2]$ we have the formulas

$$\frac{\partial F}{\partial \omega_1} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_1(\xi)d\xi,$$
$$\frac{\partial F}{\partial \omega_2} = \int_a^b F'[\varphi_0 + \omega_1\psi_1 + \omega_2\psi_2 \mid \xi]\psi_2(\xi)d\xi.$$

On account of the uniform continuity of F' over ξ and the two-parameter family of curves defined by ω_1 and ω_2, it follows that $\partial F/\partial\omega_1$ and $\partial F/\partial\omega_2$ are continuous functions of ω_1 and ω_2. Hence if we put $\omega_1 = \omega_2 = \omega$, we find that $dF/d\omega$ exists and is given by

$$\frac{dF}{d\omega} = \left(\frac{\partial F}{\partial \omega_1}\right)_{\substack{\omega_1=\omega\\ \omega_2=\omega}} + \left(\frac{\partial F}{\partial \omega_2}\right)_{\substack{\omega_1=\omega\\ \omega_2=\omega}},$$

which reduces to the form (9), since $\psi_1 + \psi_2 = \psi$. For $\omega = 0$, equation (9) reduces to equation (8), which holds for any continuous function $\psi(x)$, and is equivalent to (2).

From (9) we obtain (with Volterra) another law of the mean in the form

$$(10)\quad F[\varphi(x) + \psi(x)] - F[\varphi(x)] = \int_a^b F'[\varphi(x) + \theta\psi(x) \mid \xi]\psi(\xi)d\xi,$$

where $0 < \theta < 1$.

II. Other Methods of Deducing Formula (2).

9. We may obtain the formula (8) and hence the formula (2) from more general points of view. One point of view, which we shall consider, is closely related to the ideas which have just been developed; a second is a modification of the procedure of Fréchet.

For simplicity let us replace (α) by a new hypothesis (α'), the difference being that F is defined not only for all continuous functions in the given domain, but also for all functions in the given domain which have merely a finite number of discontinuities.*

* It is sufficient for what follows if we admit merely discontinuities *of the first kind*, so called (Lebesgue).

Let $\psi(x)$ be a continuous function, *of one sign*, and construct the functions ${}_\alpha\psi_\beta(x)$ defined as follows:

$$\begin{aligned} {}_\alpha\psi_\beta(x) &= 0 \quad (a \leqq x < \alpha,\ \beta < x \leqq b) \\ &= \psi(x) \qquad\qquad (\alpha \leqq x \leqq \beta). \end{aligned}$$

Let us consider besides (α'), the following postulates (β), (γ):

$$(\beta) \qquad \lim_{\omega=0,\ \alpha=a',\ \beta=b'} \frac{\Delta F}{\sigma} \text{ exists} \quad (a \leqq a' \leqq b' \leqq b),$$

where

$$\Delta F = F[\varphi(x) + \omega{}_\alpha\psi_\beta(x)] - F[\varphi(x)]$$

$$\sigma = \omega \int_\alpha^\beta {}_\alpha\psi_\beta(x)dx.$$

This limit will depend on $\varphi(x)$, $\psi(x)$, a' and b'; let us call it $G[\varphi(x), \psi(x) \mid a', b']$.

(γ) $\qquad G[\varphi(x), \psi(x) \mid a', b']$ is continuous in $\varphi(x)$.

10. If (α') and (β) are satisfied, we can verify directly that when a' and b' are equal, the function G is independent of $\psi(x)$, and we can therefore write

$$G[\varphi(x), \psi(x) \mid a', a'] = G[\varphi(x) \mid a'].$$

If F happens to have a functional derivative F', then

$$(11) \qquad G[\varphi(x) \mid a'] = F'[\varphi(x) \mid a'].$$

If (α') and (β) are satisfied, we see also that if we let $|\alpha - a'|$, $|b' - \beta|$ approach zero with ω as functions of ω, then

$$\begin{aligned} \left(\frac{d}{d\omega} F[\varphi(x) + \omega\,{}_\alpha\psi_\beta(x)]\right)_{\omega=0} &= G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx \qquad \text{if } a' \neq b', \text{ and} \\ &= 0 \qquad\qquad \text{if } a' = b'. \end{aligned}$$

11. If (α'), (β), (γ) are satisfied, we can deduce, by making use of partial derivatives as in § 8, that

$$(13) \qquad \begin{aligned} &G[\varphi(x), \psi(x) \mid a', b'] \int_{a'}^{b'} \psi(x)dx + G[\varphi(x), \psi(x) \mid b', c'] \\ &\qquad \times \int_{b'}^{c'} \psi(x)dx = G[\varphi(x), \psi(x) \mid a', c'] \int_{a'}^{c'} \psi(x)dx. \end{aligned}$$

And from this and the definition of $G[\varphi(x) \mid a']$ it follows that $G[\varphi(x), \psi(x) \mid a', b']$ is a continuous function of its two arguments a', b' in the region $a \leqq a' \leqq b' \leqq b$, and is hence uniformly continuous with respect to them. But these properties lead us to a formula equivalent to (8). For we have

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = G[\varphi(x), \psi(x) \mid a, b] \int_a^b \psi(x)dx,$$

and if we split the right-hand member into n parts, according to (13), and take the limit as n becomes infinite, remembering that G is uniformly continuous in a', b', we obtain the result

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = \int_a^b G[\varphi(x) \mid \xi]\psi(\xi)d\xi, \tag{14}$$

where $\psi(x)$ is any continuous function which does not change sign. This again is generalized to hold for any continuous function $\psi(x)$, by the method of § 8.* Therefore, *formula* (14), *where* $\psi(x)$ *is any function continuous* $a \leqq x \leqq b$, *is a consequence of postulates* (α'), (β), (γ).†

12. It is now evident what sort of restriction is sufficient in order that formula (1) may reduce to (2). In fact we can deduce (14) from (α') and the two following hypotheses:

(β') Let $\psi(x)$ be any limited function, continuous except for a finite number of discontinuities (of the first kind); then we assume that

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0}$$

exists and is distributive with respect to ψ, i. e.,—

$$\left(\frac{d}{d\omega} F[\varphi + \omega(\psi_1 + \psi_2)]\right)_{\omega=0} = \left(\frac{d}{d\omega} F[\varphi + \omega\psi_1]\right)_{\omega=0} + \left(\frac{d}{d\omega} F[\varphi + \omega\psi_2]\right)_{\omega=0}$$

---

* If in the definition of $G[\varphi(x), \psi(x) \mid a', b']$ we do not allow ω to change sign as it approaches zero, we are led to two functions $G_+[\varphi(x) \mid \xi]$ and $G_-[\varphi(x) \mid \xi]$, whence, instead of (14) for the general ψ, we have

$$\left(\frac{d}{d\omega} F[\varphi + \omega\psi]\right)_{\omega=0} = \frac{1}{2}\int_a^b \{G_+[\varphi(x) \mid \xi](\psi(\xi) + \mid \psi(\xi) \mid) + G_-[\varphi(x) \mid \xi](\psi(\xi) - \mid \psi(\xi) \mid)\}d\xi. \tag{14'}$$

† If $\varphi(x) = \Phi_1(x)$, $\psi(x)$ cannot be negative; if $\varphi(x) = \Phi_2(x)$, $\psi(x)$ cannot be positive.

(γ') The function $G[\varphi(x) \mid \xi]$ defined in (β) exists for $a \leqq \xi \leqq b$.

The postulate (β') is substantially the hypothesis used by Fréchet with (α) in defining the existence of a differential.* Moreover if we restrict ourselves to a function $\bar{\psi}(x)$ of one sign, we see that the function

$$G'[\varphi(x), \psi(x) \mid a', b'] = \lim_{\omega=0} \frac{F[\varphi(x) + \omega_{a'}\psi_{b'}(x)] - F[\varphi(x)]}{\omega \int_{a'}^{b'} \psi(x)dx}$$

exists, provided that $a' \neq b'$, and from (β') we see that the relation (13) holds with G' for G. And if we define

$$G'[\varphi(x), \psi(x) \mid a'a'] = G[\varphi(x), a'],$$

it follows as in (11) that G' is uniformly continuous in a' and b' in the region $a \leqq a' \leqq b' \leqq b$. Hence we obtain again formula (14) for a function ψ of one sign, and, finally, for any function ψ if ψ is limited and has merely a finite number of discontinuities (of the first kind).

13. The advantage of the last set of assumptions (α'), (β'), (γ') is that they do not demand the continuity of $G[\varphi(x) \mid \xi]$, or $F'[\varphi(x) \mid \xi]$ if it exists, with respect to the functional argument $\varphi(x)$. If for a given $\varphi(x)$, we suppose that $G[\varphi(x) \mid \xi]$ does not exist for certain special values of ξ, we have the case that F depends in a special manner on $\varphi(x)$ for those values of x.† The case where $A[\varphi(x) \mid \xi]$ in (3) is discontinuous comes under this specification. The property of possessing a functional derivative in general, however, seems to depend on fundamental properties of continuity with respect to aggregates of functions, the study of which is thereby rendered specially inviting.

THE RICE INSTITUTE,
February, 1915.

* M. Fréchet, loc. cit., p. 141. In order to justify the substitution of a postulate akin to (α') rather than (α) see F. Riesz, "Les opérations fonctionelles linéaires," *Annales scientifiques de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 2, who shows that a linear relation such as that expressed in the conception of differential can always be extended by definition to apply to certain classes of discontinuous arguments.

† V. Volterra, loc. cit., p. 144.