A CERTAIN CLASS OF FUNCTIONS CONNECTED WITH FUCHSIAN GROUPS.

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1. CONSIDER a Fuchsian group $\Gamma$ of linear substitutions

\[ V_i \equiv z_i = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} \quad (i = 1, 2, 3, \ldots) \]

\[ \alpha_i \delta_i - \beta_i \gamma_i = 1, \]

that transform the unit circle into itself, and for which the unit circle is a natural boundary. The index $i$ for which $z_i$ approaches a point of the boundary we denote by $\infty$, so that $\lim_{i=\infty} (z_i) = e^{i\phi}$, where $\phi$ may have any value from 0 to $2\pi$.

Let $z_0 = z$ represent identity. Denote by $R_0 = R$ the fundamental region in which $z$ lies, and by $R_1, R_2, \cdots$ the regions resulting from $R$ by the substitutions $V_i \quad (i = 1, 2, 3, \ldots)$. Let $e_i$ be the greatest "elongation" of the boundary of $R_i$, i.e., the maximum distance between two points of the boundary of $R_i$; then, according to a theorem due to Bricard,* it is possible to circumscribe a circle $C_i$ to the region $R_i$, such that its radius does not need to be greater than at most $e_i/\sqrt{3}$.

For $i = \infty$, the area $A_i$ of $R_i$, being that of a singly connected region bounded by circular arcs, is finite, so that for the ratio of the area of the circle $C_i$ to that of the region $R_i$ we have

\[ 1 < \frac{\pi e_i^2}{3A_i} < M \quad (i = 1, 2, 3, \ldots), \]

where $M$ is a positive finite quantity $> 1$. But it can be shown that this inequality also exists when $\lim_{i=\infty} (z_i) = e^{i\phi}$. Hence from (2) we get

\[ 3\Sigma A_i < \Sigma \pi e_i^2 < 3M\Sigma A_i, \]

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in which the sums are extended over the whole group $\Gamma$. 
as $\sum A_i = \pi$ is a finite quantity we find that the sum of the areas of all circles $C_i$, and consequently the sum of the squares of the radii of all these circles is finite.

2. Choose now within $R$ any two points $a$ and $b$ and a variable point $z$, so that the area formed by the euclidean triangle $z_i a_i b_i$ lies entirely within $C_i$. Now

$$\left| z_i - a_i \right| \leq e_i; \quad \left| z_i - b_i \right| \leq e_i,$$

hence

$$\left| z_i - a_i \right| \cdot \left| z_i - b_i \right| \leq e_i^2,$$

and

$$\sum_{i=0}^{\infty} \left| z_i - a_i \right| \cdot \left| z_i - b_i \right| \leq \sum_{i=0}^{\infty} e_i^2.$$

But, according to (3), $\sum e_i^2$ is a finite quantity. The left side of (4) is therefore an absolutely convergent series, for all values of $z$ within $R$. The condition for uniform convergence within the whole domain is evidently also satisfied, so that we can state

**Theorem I.** The series

$$\sum_{i=0}^{\infty} (z_i - a_i)(z_i - b_i)$$

extended over a Fuchsian group $\Gamma$, with the unit circle as a natural boundary and $z, a, b$ lying within the fundamental region of $\Gamma$, is a uniformly convergent series, and defines an analytic function within $R$ that vanishes for $z = a$ and $z = b$ and has no infinities within $R$. The result is still valid when $z_b = z_a$, so that

$$\sum_{i=0}^{\infty} (z_i - a_i)^2$$

also defines such a function which at $z = a$ has a zero of the second order.

3. The theorem may immediately be generalized. Choose for $z$ and $a$ any two points within the unit circle (excluding the boundary). The straight line joining them is cut by a finite number of polygons $R_i$ into the segments $l_1, l_2, l_3, \ldots, l_r$.

Any substitution $V_\lambda \equiv \begin{pmatrix} \alpha_\lambda \beta_\lambda \\ \gamma_\lambda \delta_\lambda \end{pmatrix}$ of the group $\Gamma$ transforms the
straight segment from $z$ to $a$ into an arc of a circle from $z_\lambda$ to $a_\lambda$ and the segments $l_\iota$ into arcs $l_{i\lambda}$ intercepted by the corresponding polygons arising from the substitution $V_\lambda$. Every arc $l_{i\lambda}$ is subtended by a chord $s_{i\lambda} < l_{i\lambda} < e_{i\lambda}$, where $e_{i\lambda}$ denotes the elongation of the polygon (region) $R_{i\lambda}$. From this follows immediately that

$$f_\lambda = |z_\lambda - a_\lambda| < e_{1\lambda} + e_{2\lambda} + \cdots + e_{r\lambda},$$

and

$$(f_\lambda)^2 < (e_{1\lambda} + e_{2\lambda} + \cdots + e_{i\lambda} + \cdots + e_{k\lambda} + \cdots + e_{r\lambda})^2.$$ 

From the inequality

$$2e_{i\lambda}e_{k\lambda} < e_{i\lambda}^2 + e_{k\lambda}^2,$$

we derive without difficulty

$$(5) \quad 2 \sum_{\iota=1}^{r} e_{i\lambda}e_{k\lambda} < (r-1)(e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2).$$

Now

$$\sum_{\lambda=0}^{\infty} (f_\lambda)^2 = \sum_{\lambda=0}^{\infty} (e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2) + 2 \sum_{\lambda=0}^{\infty} \sum_{\iota=1}^{r} e_{i\lambda}e_{k\lambda};$$

hence, according to (5),

$$(6) \quad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r \sum_{\lambda=0}^{\infty} (e_{1\lambda}^2 + e_{2\lambda}^2 + \cdots + e_{r\lambda}^2).$$

But

$$\sum_{\lambda=0}^{\infty} e_{i\lambda}^2 = \sum_{\lambda=0}^{\infty} e_{k\lambda}^2,$$

so that (6) reduces to

$$(7) \quad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r^2 \sum_{\lambda=0}^{\infty} e_{\lambda}^2.$$ 

The right side of this inequality is a finite quantity, so that the series on the left side is absolutely convergent.

Hence

**Theorem II.** The series

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)^2$$

...
extended over a Fuchsian group with the unit circle as a natural
boundary, where $z$ and $a$ are any two points within the unit circle
and not on the boundary, when $a$ is fixed, is an absolutely and
uniformly convergent series of $z$ for all points within and not on
the boundary, and represents an analytic function in the neighbor­
hood of all such points. It has a zero of the second order for
$z = a$, and has the unit circle as a natural boundary.

4. This theorem admits of a further generalization. Choose
any three points $z, z', a$ within and not on the unit circle, and
write $f_\lambda = |z_\lambda - a_\lambda|$, $g_\lambda = |z'_\lambda - a_\lambda|$. Assuming $f \neq 0$
and $g \neq 0$, it is possible to find a positive finite number $M$
such that the ratio $g_\lambda/f_\lambda < M$, $\lambda = 1, 2, 3, \ldots$, also when $z_\lambda$
approaches a point on the unit circle. We have therefore
$g_\lambda < Mf_\lambda$, and

$$f_\lambda g_\lambda < Mf_\lambda^2,$$

and consequently

$$\sum_{\lambda=0}^\infty f_\lambda g_\lambda < M \sum_{\lambda=0}^\infty f_\lambda^2.$$ (8)

As the right side of this inequality is absolutely convergent,

$$\sum_{\lambda=0}^\infty f_\lambda g_\lambda$$

is an absolutely convergent series, and that consequently

$$\sum_{\lambda=0}^\infty (z_\lambda - a_\lambda)(z'_\lambda - a_\lambda)$$

is absolutely and uniformly convergent, and, for $a$ and $z'$
constant, defines an analytic function of $z$ for all points
within and not on the boundary of the unit circle. It vanishes
for $z = a$ and has the unit circle as a natural boundary.

Nothing is lost in the convergency proof of (9) by assuming
$z$ and $z'$ fixed and $a$ as variable. Hence putting in (9) $z = a,$
$z' = b$ and $a = z$ we may state

**Theorem III.** The series

$$\sum_{\lambda=0}^\infty (z_\lambda - a_\lambda)(z_\lambda - b_\lambda),$$

where $a$ and $b$ are any two points within and not on the unit circle,
is absolutely and uniformly convergent and represents an analytic
function of \( z \) within the unit circle, which is a natural boundary of the function. It has \( z = a \) and \( z = b \) as zeros.

5. Making use of the proposition that for an analytic function \( F(z) \) which within a certain region has the character of a rational function, such that for any point \( z_0 \) of this region \( F(z_0) \) exists,

\[
\lim_{z \to z'} \left( \frac{F(z) - F(z')}{z - z'} \right) = F'(z_0)
\]

we may extend theorem III to an even more general type of functions. Let \( \mathcal{R}(z) \) be a rational function of \( z \) which for \( z = 0 \) does not become infinite. Putting \((z_\lambda - a_\lambda)(z_\lambda - b_\lambda) = u_\lambda, (z'_\lambda - a'_\lambda)(z'_\lambda - b'_\lambda) = u'_\lambda\), where \( z', a', b' \) denote a set like \( z, a, b \), then as \( z_\lambda, a_\lambda, b_\lambda, u_\lambda, u'_\lambda \) will approach the same point, and \( u \) and \( u' \) will approach zero as a limit. Consequently

\[
\lim_{z_\lambda \to e^i\phi} \left( \frac{\mathcal{R}(z_\lambda) - \mathcal{R}(u_\lambda)}{u_\lambda - u'_\lambda} \right) = \mathcal{R}'(0)
\]

is a finite quantity, and as \( \sum (u - u') \) is absolutely and uniformly convergent, also \( \sum_{\lambda=0}^{\infty} \{\mathcal{R}(u) - \mathcal{R}(u')\} \) will be absolutely and uniformly convergent within the unit circle, except for a finite number of values of \( u \) and \( u' \), which are poles of \( \mathcal{R}(u) \), and their congruents in the group \( \Gamma \). Hence, with the expressions for \( u, u' \) and \( \mathcal{R} \) defined as above, we may state

**Theorem IV.** When \( a, b, a', b', z' \) are fixed, so that no \( u'_\lambda \) is a pole of \( \mathcal{R}(z) \), then

\[
\sum_{\lambda=0}^{\infty} \left[ \mathcal{R}(z_\lambda - a_\lambda)(z_\lambda - b_\lambda) - \mathcal{R}(z'_\lambda - a'_\lambda)(z'_\lambda - b'_\lambda) \right]
\]

extended over the whole Fuchsian group represents an analytic function of \( z \), which has the same poles as those of \( \mathcal{R}(x) \) and their congruents, and which has the unit circle as a natural boundary.

It appears that in general these functions are not automorphic in the ordinary sense.

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* For a statement of formula (10) and its applications to trigonometric and elliptic functions see Schottky: "Ueber die Funktionenklasse, die der Gleichung \( F \left( \frac{ax + b}{\gamma x + \delta} \right) = F(x) \) genügt"; Crelle, vol. 143 (1913), pp. 1–24.