so that
\[ \sum_{i=1}^{\infty} \beta_i \leq c. \]

The points of \( E \) are now enclosed in the open intervals \( \alpha_{ij} \),
so that each point is inside of an infinite number of intervals,
and \( \phi(x) \) is defined to be the sum of all the \( \alpha \)-intervals or parts thereof which lie to the left of \( x \).

Thus \( \phi(x) \) is monotone and can easily be shown to be absolutely continuous as follows:

If \( i + j = N \) is chosen sufficiently large, the \( \phi_N(x) \) formed
for this finite set of intervals will be absolutely continuous
and as near as we please to \( \phi(x) \) for all values of \( x \). Hence \( \phi(x) \) is absolutely continuous.

If the set \( E \) is not an inner limiting set, the set \( E'' = E + \overline{E} \), which lies inside an infinite number of \( \alpha \) intervals,
will be such a set, and \( \phi(x) \) will have an infinite derivative at
all the points of \( E'' \) and no others. The set \( E \) may itself be
an inner limiting set, in which case \( \overline{E} = 0 \).

It would be interesting to determine whether all absolutely
continuous functions are of the form

\[ F(x) + \phi(x), \]

where \( F(x) \) has limited derivates.

AUSTIN, TEXAS.

ON THE REPRESENTATION OF NUMBERS IN THE
FORM \( x^3 + y^3 + z^3 - 3xyz \).

BY PROFESSOR R. D. CARMCHEAL.

(Read before the American Mathematical Society, August 3, 1915.)

If by \( g(x, y, z) \) we denote the form

(1) \[ g(x, y, z) = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx), \]

then it is well known that

\[ g(x, y, z) \cdot g(u, v, w) = g(xu + yv + zw, xu + yu + zw, \]
\[ xw + yv + zu). \]
By interchanging the roles of $v$ and $w$, we also have
\[ g(x, y, z) \cdot g(u, v, w) = g(xu + yv + zw, xv + yu + vz, xv + yw + zu). \]

Obviously these two representations of the product are identical if $v = w$. Since $g$ is a symmetric function of its arguments it is easy to see that they are identical in each of the following six cases: $v = w$, $v = u$, $w = u$, $x = y$, $x = z$, $y = z$. On the other hand if we assume that the two representations are identical we are led to one of the preceding six equalities. Thus we have the following theorem:* 

**Theorem I.** If $r, s, t$ have either of the two sets of values $(r_1, s_1, t_1)$ and $(r_2, s_2, t_2)$, where

\[
\begin{align*}
(1) & \quad r_1 = xu + yw + vz, \quad r_2 = xu + yv + zw, \\
(2) & \quad s_1 = xv + yu + zw, \quad s_2 = xv + yu + vz, \\
& \quad t_1 = xw + yv + zu, \quad t_2 = xv + yw + zu,
\end{align*}
\]

then
\[ g(x, y, z) \cdot g(u, v, w) = g(r, s, t). \]

In order that the two expressions $g(r, s, t)$ shall be non-identical it is necessary and sufficient that each of the two sets $(x, y, z)$ and $(u, v, w)$ shall consist of distinct members.

It may be observed that for each set of values $(r, s, t)$ we have
\[ r + s + t = (x + y + z)(u + v + w). \]

If $a$ and $b$ are both representable in the form $g$, then the product $ab$ is representable in the same form, as is seen from the foregoing theorem. The question arises as to whether all the representations of $ab$ are obtained by means of Theorem I from the representations of $a$ and $b$. That this is to be answered in the negative follows from the simplest examples. Thus it is easy to show that 2 is represented in the form $g$ in only one way, namely, $2 = g(1, 1, 0)$. From this and Theorem I we have $4 = g(2, 1, 1)$, the two sets $(r, s, t)$ being equivalent in this case. But we have also $4 = g(1, 1, -1)$. That is, 4 is capable of a representation in the form $g$ not obtainable by means of Theorem I from the representation of its proper factors.

* The result in this theorem is well known, as we have just pointed out. The remaining theorems in the paper are believed to be new.
From these two representations of 4 it follows that a number may be represented in two ways by the form \( g \) and yet these representations not result from writing the product of its factors in two ways in the form \( g \) by means of Theorem I. Two other examples illustrating this are afforded by the following relations: \( 20 = g(3, 1, 1) = g(7, 7, 6); \ 91 = g(6, 4, 3) = g(31, 30, 30) \).

We observe that if the numbers \( x, y, z, u, v, w \) in Theorem I are all non-negative then \( r, s, t \) are likewise non-negative. This leads us to consider the problem of the representation of numbers in the form \( g \) when the arguments are restricted to be non-negative. The fundamental theorem here is the following:

**Theorem II.** Every prime number \( p \) other than 3 is representable in one way and in only one way in the form

\[
p = g(x, y, z) = x^3 + y^3 + z^3 - 3xyz,
\]

where the arguments \( x, y, z \) are restricted to be non-negative.

In order to prove this let us seek to put \( p \) in the form

\[
p = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).
\]

Since the numbers \( x, y, z \) are to be non-negative it is clear that this equation can be satisfied only when

\[
x + y + z = p, \ x^2 + y^2 + z^2 - xy - yz - zx = 1.
\]

Without loss of generality we may assume that \( x \geq y \geq z \), and this we do. Let us write

\[
x = u + z, \ y = v + z.
\]

Then \( u, v, \) and \( u - v \) are non-negative numbers. Equations (4) may now be written

\[
3z + u + v = p, \ u^2 - uv + v^2 = 1.
\]

From the latter equation we have \( (u - v)^2 + uv = 1 \). From this it follows that \( u = v = 1 \) or \( u = 1, \ v = 0 \). From the first equation in (5) we see that the former set must be used when \( p \) is of the form \( 3k + 2 \) and the latter when \( p \) is of the form \( 3k + 1 \), in order that \( z \) shall be an integer. In either case \( u, v, z, \) and therefore \( x, y, z \), are uniquely determined. Hence the theorem.

Now \( g(2, 1, 0) = 9 \). From this fact and Theorems I and II
it follows that every positive number is representable in the form \( g(x, y, z) \) with non-negative arguments with the possible exception of those of the form \( 3t \), where \( t \) is not divisible by 3. Now, we have

\[
g(x, y, z) = (x + y + z) \{ (x + y + z)^2 - 3(xy + xz + yz) \}
\]

If the second member of this equation is divisible by 3, so is \( x + y + z \), and therefore this second member is divisible by 9 (whatever signs \( x, y, z \) may have). Hence the form \( g(x, y, z) \) does not contain any number \( 3t \) where \( t \) is an integer prime to 3. Thence we have the following theorem:

**Theorem III.** The positive integers which may be represented in the form \( g(x, y, z) \) include all positive integers with the sole exception of those which are divisible by 3 but not by 9. In every case the arguments \( x, y, z \) in the representation may be chosen so as to be all non-negative.

If \( x, y, z \) are allowed to be negative it is no longer true that primes are always uniquely represented in the form \( g(x, y, z) \). Thus we have 7 = \( g(3, 2, 2) = g(2, -1, 0) \), 13 = \( g(5, 4, 4) = g(2, -2, 1) \). Then let us consider more generally the representation of a prime \( p \) in the form

\[
p = g(x, y, z) = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz).
\]

Writing \( x = u + z, y = v + z \), we have

\[
(6) \quad p = (3z + u + v)(u^2 - uv + v^2).
\]

Now \( 4(u^2 - uv + v^2) = (u + v)^2 + 3(u - v)^2 \), so that \( u^2 - uv + v^2 \) is not negative. Hence from (6) it follows that this expression has the value 1 or the value \( p \). Therefore we have to examine the following two cases:

(a) \( u^2 - uv + v^2 = 1 \), \( 3z + u + v = p \);

(b) \( u^2 - uv + v^2 = p \), \( 3z + u + v = 1 \).

Now the equation \( u^2 - uv + v^2 = 1 \), or \((u + v)^2 + 3(u - v)^2 = 4\), has only the solutions obtained in the proof of Theorem II. Hence case (a) gives rise only to the representation by means of non-negative arguments \( x, y, z \) treated in Theorem II.

Let us next consider case (b). We have

\[
(7) \quad 4p = (u + v)^2 + 3(u - v)^2.
\]
Since \( p \neq 3 \) it follows from this that \( p \equiv 1 \mod 3 \), so that \( p \) is of the form \( 6n + 1 \). Equation (7) has a solution for every prime \( p \) of the form \( 6n + 1 \) such that \( u + v \equiv 1 \mod 3 \).* Furthermore \( u + v \) and \( u - v \) are obviously both odd or both even, so that \( u \) and \( v \) are themselves integers. From the second equation in (b) it follows now that \( z \), and hence \( x \) and \( y \), are integers. Thus we have a representation of \( p \) in the desired form \( g(x, y, z) \), one at least of the arguments \( x, y, z \) being obviously negative. Furthermore it is clear that this representation is unique provided that \( 4p \) has only one representation

\[ 4p = a^2 + 3b^2, \quad a > 0, \quad b > 0, \]

in which \( a \equiv 1 \mod 3 \), since \( u + v \) must have a value congruent to unity modulo 3 in order that \( z \) shall be an integer. This latter fact concerning \( 4p \) we shall now prove. Let \( 4p \) have the representation

\[ 4p = \alpha^2 + 3\beta^2, \quad \alpha > 0, \quad \beta > 0. \]

Then we have

\[
(8) \quad 16p^2 = (a\alpha + 3b\beta)^2 + 3(a\beta - ab)^2 = (a\alpha - 3b\beta)^2 \\
+ 3(a\beta + ab)^2
\]

and

\[
(9) \quad 4p(a^2 - \alpha^2) = 3(\alpha b + a\beta)(\alpha b - a\beta).
\]

Hence \( p \) is a factor of \( ab + a\beta \) or of \( ab - a\beta \). Suppose that \( p \) is a factor of \( ab + a\beta \), the complementary factor being \( s \). Then from (8) it follows that \( p \) is a factor of \( a\alpha - 3b\beta \); let the complementary factor be \( t \). Then from (8) we have

\[ 16 = t^2 + 3s^2; \]

whence \( t = 4, s = 0 \) or \( t = s = 2 \). If the former solution is taken, we find from (9) that \( a = \alpha \) and hence that the two representations of \( 4p \) are identical. If we take the latter we have

\[ ab + a\beta = 2p, \quad a\alpha - 3b\beta = 2p; \]

whence it follows readily that

\[ 2\alpha = a + 3b. \]

* See Bachmann's Kreistheilung, pp. 138–141.
Since $a \equiv 1 \mod 3$ it follows that $a \equiv 2 \mod 3$. In a similar way one may treat the case when $ab - a\beta$ is divisible by $p$ and with a similar result. Therefore $4p$ can be represented in the form $a^2 + 3b^2$ in only one way provided that $a$ is restricted to be congruent to unity modulo $3$.

We are thus led to the following theorem:

**Theorem IV.** A prime number $p$ of the form $6n + 1$ may be represented in one and in only one way in the form

$$p = g(x, y, z) = x^3 + y^3 + z^3 - 3xyz$$

where one at least of the arguments $x, y, z$ is negative. No other prime number has such a representation. (Compare Theorem II.)

Let us next consider the representation of $p^2$ in the form $g(x, y, z)$, $p$ being a prime number different from $3$.

Writing $x = u + z, y = v + z$, we have

$$p^2 = (3z + u + v)(u^2 - uv + v^2).$$

Since $u^2 - uv + v^2$ cannot be negative it follows that there are three cases to be examined, namely:

(a) $3z + u + v = p^2, u^2 - uv + v^2 = 1$;

(b) $3z + u + v = p, u^2 - uv + v^2 = p$;

(c) $3z + u + v = 1, u^2 - uv + v^2 = p^2$.

These may be treated by the methods already employed. We take up the cases in order.

The second equation in (a) has the two solutions $u = v = 1; u = 1, v = 0$, and no others (if we take $u \geq v$, as we may without loss of generality). Since $z$ must be integral it follows from the first equation in (a) that we must take $u = 1, v = 0$. We are thus led to the following conclusion:

There is a unique representation of $p^2 (p + 3)$ in the form $g(x, y, z)$ subject to the condition $x + y + z = p^2$. In case (b) it is easy to show from the second equation that $p$ is of the form $6n + 1$. Proceeding as in the proof of Theorem

* As a corollary of this argument we have the following result:

If $p$ is a prime number of the form $6n + 1$ then $4p$ can be represented in two and in only two ways in the form $a^2 + 3b^2$, $a$ and $b$ being positive, and in one of these ways $a$ is congruent to $1$ and in the other $a$ is congruent to $2$ modulo $3$.

† For the excluded case we have $9 = g(2, 1, 0)$. 

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IV, we find that there is a unique solution of equations (b) subject to the condition that \( z \) is integral. We thus conclude:

In order that \( p^2 \ (p \neq 3) \) shall be representable in the form \( g(x, y, z) \), with the condition \( x + y + z = p \), it is necessary and sufficient that \( p \) be of the form \( 6n + 1 \) and this representation, when it exists, is unique.

In case (c) the second equation has the obvious solution \( u = v = p \). This solution will yield integral \( z \) only when \( p \) has the form \( 3k + 2 \). The solution is unique for such \( p \) since it follows from the theory of binary quadratic forms that such a prime power \( p^2 \) can be represented in the form \( u^2 - uv + v^2 \) only when \( u = v = p \) or \( u = p, v = 0 \), the latter solution giving \( z \) non-integral in the present case. If \( p \) is of the form \( 3k + 1 \) then the second equation in (c) has the solution \( u = p, v = 0 \); this gives rise to integral \( z \) and hence to a representation of the kind sought. The representation in this case is not necessarily unique, since the second equation in (c) may have a second solution giving rise to integral \( z \). We have the following result:

The prime power \( p^2 \ (p \neq 3) \) can be represented in the form \( g(x, y, z) \) subject to the condition \( x + y + z = 1 \).

ON THE LINEAR CONTINUUM.

BY DR. ROBERT L. MOORE.

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§ 1. Introduction.

In the Annals of Mathematics, volume 16 (1915), pages 123–133, I proposed a set \( G \) of eight axioms for the linear continuum in terms of point and limit. Betweenness was defined,* and it was stated that the set \( G \) is categorical with respect to point and the thus defined betweenness.† In the present paper it is shown that, although this statement is true, nevertheless

* See Definition 3, loc. cit., p. 125.

† This statement, which is proved in the present paper, implies that if \( K \) is any statement in terms of point and betweenness, then either it follows from Axioms 1–8 and Definition 3 that \( K \) is true or it follows from Axioms 1–8 and Definition 3 that \( K \) is false.