IV, we find that there is a unique solution of equations (b) subject to the condition that z is integral. We thus conclude:

In order that p^2 $(p \neq 3)$ *shall be representable in the form* $g(x, y, z)$, *with the condition* $x + y + z = p$, *it is necessary and sufficient that* p *be of the form* $6n + 1$ *and this representation, when it exists, is unique.*

In case (c) the second equation has the obvious solution $u = v = p$. This solution will yield integral z only when p has the form $3k + 2$. The solution is unique for such p since it follows from the theory of binary quadratic forms that such a prime power p^2 can be represented in the form $u^2 - uv + v^2$ only when $u = v = p$ or $u = p, v = 0$, the latter solution giving z non-integral in the present case. If p is of the form $3k + 1$ then the second equation in (c) has the solution $u = p$, $v = 0$; this gives rise to integral z and hence to a representation of the kind sought. The representation in this case is not necessarily unique, since the second equation in (c) may have a second solution giving rise to integral z. We have the following result:

The prime power p^2 $(p \neq 3)$ *can be represented in the form* $g(x, y, z)$ *subject to the condition* $x + y + z = 1$.

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ON THE LINEAR CONTINUUM.

BY DR. ROBERT L. MOORE.

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§ 1. *Introduction.*

In the *Annals of Mathematics,* volume 16 (1915), pages 123–133, I proposed a set G of eight axioms for the linear continuum in terms of *point* and *limit*. Betweenness was defined,* and it was stated that the set G is categorical with respect to *point* and *the thus defined betweenness*.† In the present paper it is shown that, although this statement is true, nevertheless

* See Definition 3, loc. cit., p. 125.

† This statement, which is proved in the present paper, implies that if K is any statement in terms of point and betweenness, then either it follows from Axioms 1–8 and Definition 3 that K is true or it follows from Axioms 1–8 and Definition 3 that K is false.

G is not *absolutely* categorical,* that is to say it is not categorical with respect to *point* and *limit*, the undefined symbols in terms of which it is stated.

An absolutely categorical set is obtained if Axiom 5 is replaced by the following axiom.

AXIOM 5′. *If r_1 and r_2 are two mutually exclusive, non-complementary rays,† then every infinite set of points lying in $S - (r_1 + r_2)$ has at least one limit point.*

§ 2. *On the Non-Categoricity of the Set G.*

That G is not categorical is shown by the existence of the following examples E and $E_{5'}$. The letter K will be used to denote the statement that the point P is a *limit point* of the point set M whenever every segment containing P contains at least one point of M distinct from P.

$E_{5'}$. Let the space S be an ordinary linear continuum $(0 < x < 1)$ but interpret the statement that P is a *limit point* of M to mean that P is a limit point in the ordinary sense of a rational subset of M. Here Axioms 1–8 are satisfied‡ but statement K is false.

E. Let S be an ordinary linear continuum and let *limit*

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* For definition of categoricity (in the absolute sense) see O. Veblen. "A system of axioms for geometry," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 343–384.

† If P is a point and S is the set of all points, and $S - P = S_P' + S_P''$, where S_P' and S_P'' are mutually exclusive connected point sets neither of which contains a limit point of the other one, then S_P' and S_P'' are called *rays*. If B is a point of the ray S_P' then the ray S_P' is denoted by PB. The ray S_P' is said to be complementary to (or the complement of) the ray S_P''.

‡ That Axiom 5 is satisfied in this example may be proved as follows. In this proof the phrase "limit point" (unitalicized) has its ordinary meaning while "*limit point*" (in italics) is to be interpreted as defined in E_5'.

Suppose that $S = K_1 + K_2$ where K_1 and K_2 are mutually exclusive point sets. There are two cases to be considered.

Case I. Suppose K_1 contains no rational subset. Then K_2 contains the set of all rational points. But every point of S is a limit point of this set. Hence every point of K_1 is a *limit point* of K_2.

Case II. Suppose $K_1 = R_1 + I_1$ and $K_2 = R_2 + I_2$, where R_1 and R_2 are composed entirely of rational points, while I_k $(k = 1, 2)$ either is vacuous or is composed entirely of irrational points. Suppose K_1 contains no limit point of R_2 and K_2 contains no limit point of R_1. Then one of the point sets I_1 and I_2 must contain a limit point of the other one. Suppose I_1 contains a limit point of I_2. Since every point of I_2 is a limit point of $R_1 + R_2$ but not of R_1, therefore every point of I_2 is a limit point of R_2. It follows that I_1 contains a limit point of R_2 and therefore K_1 contains a *limit point* of K_2.

point have its usual significance. In this example also Axioms 1–8 are all satisfied. But here the statement K is true.

From the existence of these two examples it is clear that neither K nor its contradictory is a consequence of Axioms 1–8. Hence this system of axioms is not absolutely categorical.

§ 3. *Consequences of Axioms 1–4, 6, 7.*

THEOREM A. *No point is a limit point of a finite set of points.*

Theorem A is a consequence of Axioms 2 and 3.

THEOREM B. *Every ray contains infinitely many points.*

Theorem B is a consequence of Theorem A and Theorem 1.*

THEOREM C. *Every ray contains an infinite set of points that has no limit point.*

Proof. By Axiom 7 there exists a countable set of points R such that every point either belongs to R or is a limit point of R. If the ray AB did not contain infinitely many points of R, then, by Theorem B, Axiom 2 and Theorem A, AB would contain a limit point of AB', its complement. But this is contrary to Definition 2. Hence AB and R contain infinitely many points in common. Let P_1, P_2, P_3 be the set of all such common points. By Theorem 11 there exists a point X_1 such that AP_1X_1. There exists K_1 such that AX_1K_1. By Theorem 2, AX_1 contains X_1K_1. But AX_1 is the same as AB. Thus AB contains X_1K_1. But, by Theorem 4, X_1P_1A. Therefore P_1 is on X_1A. Consequently P_1 is not on X_1K_1. Thus the ray X_1K_1 lies in AB but does not contain P_1. Similarly there exists a ray X_2K_2 lying in X_1K_1 (and therefore in AB) and not containing P_2. Continue this process, thus obtaining two sequences of points $X_1, X_2, \cdots$ and $K_1, K_2, \cdots$ such that AB contains X_nK_n, X_nK_n contains $X_{n+1}K_{n+1}$ and X_nK_n contains no point of the set $P_1, P_2, \cdots, P_n$. Suppose the infinite set of points $X_1, X_2, X_3, \cdots$ has a limit point X. The points $X_{n+1}, X_{n+2}, X_{n+3}, \cdots$ all lie on X_nK_n. Hence for every n, X lies on X_nK_n. Now A is not on X_nK_n. Hence it is not on X_nX. Therefore AX_nX is true for every n. Hence XA contains every X_n. But X_nA is the complement of X_nK_n, and therefore contains $P_1, P_2, P_3, \cdots, P_n$. Furthermore XA contains X_nA. Therefore XA contains all the points

* Arabic numerals are used for theorems and definitions contained in my paper "The linear continuum in terms of point and limit," loc. cit.

$P_1, P_2, \cdots, P_n$. But there exists a point Y such that AXY. The rays XY and XA are complementary. Therefore XY contains no P_n. But XY is a subset of AB. Therefore XY contains no point of R. Hence Y is not a limit point of R. Thus the supposition that the set of points $X_1, X_2, \cdots$ has a limit point leads to a contradiction.

§ 4. *Consequences of Axioms 1–4, 5′, 6, 7.*

THEOREM D.* *There do not exist three mutually exclusive rays.*

Theorem D is a consequence of Axiom 5′ and Theorem C.

THEOREM E. *If P is a limit point of M then every segment containing P contains at least one point of M distinct from P.*

Proof. Let AB denote a segment† containing P. There exist points C and D such that ABC and BAD. The ray BC is the complement of BA, while AD is the complement of AB. If the point X does not belong to the segment AB, then, by Theorems 14 and 4 and Definition 3, X is not common to the rays AB and BA. Hence if no point of M except P is in the segment AB then $M = M_1 + M_2$, where no point of M_1 except P is in AB and no point of M_2 is in BA. But P is in both AB and BA. Hence P is a limit point of neither M_1 nor M_2. Therefore, by Axiom 2, P is not a limit point of M. But this is contrary to hypothesis.

THEOREM F. *There exists a countable, everywhere dense‡ set of points.*

Theorem F is a consequence of Axiom 7 and Theorem E.

It follows§ from Theorems 4–14 and Theorem F that the set of Axioms 1–7|| is categorical with respect to *point* and *betweenness* as defined in Definition 3.

THEOREM G. *If every segment containing P contains at least one point of M distinct from P then P is a limit point of M.*

Proof. Between S and the linear continuum $(0 < x < 1)$ there is a one-to-one reciprocal correspondence that *preserves*

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* See Axiom 5, loc. cit., p. 126.

† The *segment* AB is the set of all points $[X]$ such that AXB.

‡ A set of points M is said to be everywhere dense if every segment contains a point of M.

§ See G. Cantor, "Zur Begründung der transfiniten Mengenlehre, I," *Mathematische Annalen*, vol. 46 (1895), p. 510.

|| It is to be noted that Theorems E and F are both consequences of Axioms 1–7 as well as of Axioms 1–4, 5′, 6, 7.

order. It follows that M contains an infinite set of points $P_1, P_2, P_3, \cdots$ such that for every segment τ containing P there exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in τ. It follows, with the help of Axiom 5′, that the set of points $P_1, P_2, P_3, \cdots$ has at least one limit point O. Suppose that O is distinct from P. Then there exist points A, B, and C in the order $APBOC$. There exists n such that $P_{n+1}, P_{n+2}, P_{n+3}, \cdots$ all lie in the segment AB. Hence not more than n points of the set $P_1 + P_2 + P_3 \cdots$ lie in the segment BC. Therefore, by Theorem E, Axiom 2 and Theorem A, O is not a limit point of $P_1 + P_2 + P_3 \cdots$. Thus the supposition that P is distinct from O leads to a contradiction. It follows that P is a limit point of $P_1 + P_2 + P_3 + \cdots$ and therefore of M.

§ 6. *Conclusion.*

THEOREM H. *The set of Axioms 1–4, 5′, 6, 7 is an absolutely categorical set of axioms for the linear continuum.*

Proof. It has been shown that this set of axioms is categorical with respect to point and betweenness as defined in Definition 3. But every statement in terms of *point* and *limit point* of a point set is,* in the presence of these axioms and Definition 3, equivalent to a statement in terms of *point* and *betweenness*. It follows that the set of Axioms 1–4, 5′, 6, 7 is categorical with respect to *point* and *limit point* of a point set.

That, in the set of Axioms 1–4, 5′, 6, 7, Axioms 2, 3, 4, 5′, and 6 are independent is shown by Examples E_2–E_6 of my paper in the *Annals*. That 1 and 7 are independent in this set is shown by the following examples, E_1 and E_7.

E_1. S is an ordinary linear continuum. The point P is a *limit point* of the point set M if and only if P is a limit point of M in the usual sense but M is not the set of all points.

E_7.† S is the set of all real number pairs (x, y) such that $0 < x < 1$ and $0 \leqq y \leqq 1$. The point (x_1, y_1) is a *limit point* of the point set M if, and only if, it is true that corresponding to each preassigned positive number ϵ there exists, in the set

* See Theorems E and G.

† The example E_7 was constructed with the assistance of an example given by Veblen in connection with his postulate of uniformity. Cf. O. Veblen, "Definition in terms of order alone in the linear continuum and in well-ordered sets," *Transactions of the American Mathematical Society*, vol. 6 (1905), p. 169.

M, a point (x, y), distinct from (x_1, y_1), such that 1) $|x - x_1| < \epsilon$, 2) $x - x_1 = 0$ and $|y - y_1| < \epsilon$, in case y_1 is distinct from 0 and from 1, 3) $x \leqq x_1$ if $y_1 = 0$, 4) $x \geqq x_1$ if $y_1 = 1$.

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A PROBLEM IN THE KINEMATICS OF A RIGID BODY.

BY PROFESSOR PETER FIELD.

THE problem of finding the acceleration of any point in a rigid body when the accelerations of three points are given, and incidentally of finding what is by this means determined regarding the velocities, has received but little attention. A theorem due to Burmeister solves the problem of finding the acceleration of any point in the plane of the three points whose accelerations are given. The theorem states: "If at four coplanar points P_1, P_2, P_3, P_4 the accelerations be drawn, their extremities lie in a plane and form a quadrilateral which is affine with the quadrilateral formed by the four points."

R. Mehmke* and J. Petersen† have considered the general case, but their results do not agree, owing to an oversight in Petersen's treatment. While their work is independent, the proof in both cases depends directly on the fact that when the distance between two points is constant the projections of their velocities on their joining line are equal and the projections of their accelerations on this line differ by $\omega^2 l$, l being the distance between the two points and ω the angular velocity of the line. The purpose of this paper is to show that the problem can be solved very simply by using the expressions for the accelerations which are ordinarily given in text books on mechanics, and by this method the kinematical meaning of the solution is also evident.

Let there be given the accelerations at three points. It is proposed to find what can be determined regarding the kinematical state of the body at the given instant. As the acceleration at any point in the plane of the three points can be

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* Festschrift zur Feier des 50jährigen Bestehens der technischen Hochschule Darmstadt, page 77.

† Kinematik, page 46.